Abstract

On the Automorphism Tower of Free Nilpotent Groups

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In this thesis I study the automorphism tower of free nilpotent groups. Our main tool in studying the automorphism tower is to embed every group as a lattice in some Lie group. Using known rigidity results the automorphism group of the discrete group can be embedded into the automorphism group of the Lie group.

This allows me to lift the description of the derivation tower of the free nilpotent Lie algebra to obtain information about the automorphism tower of the free nilpotent group.

The main result in this thesis states that the automorphism tower of the free nilpotent group $\Gamma(n,d)$ on n generators and nilpotency class d, stabilizes after finitely many steps. If the nilpotency class is small compared to the number of generators we have that the height of the automorphism tower is at most 3.

On the Automorphism Tower of Free Nilpotent Groups

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Contents

Notations

Here we list the notations, used in this diseration.

$\mathbb{Z}$	set of integres
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{F}_p$	finite field with p elements
$\mathfrak{gl}_n$	Lie lagebra of $n \times n$ matrices
$\mathfrak{sl}_n$	Lie lagebra of traceless $n \times n$ matrices
$\mathbb{R}\langle V \rangle$	free associative algebra over $\mathbb{R}$ generated by a spcase V
$L[V]$	free Lie algebra generated by a spcase V
$L[V,d]$	free nilpotent of class d Lie algebra generated by a spcase V
$\operatorname{Der} L$	derivation algebra of the algebra L
$\operatorname{Der}_0 L$	algebra restricted derivations of the algebra L
$\operatorname{Der}^k L$	k-th algebra in the derivation tower of L
$\operatorname{Der}_0^k L$	k-th algebra in the tower of restricted derivations of L
$\operatorname{Cen}(L)$	center of the algebra L
$\operatorname{Cen}_L(I)$	centralizer of the ideal I in the algebra L
$[A,B]$	commutator of the idales A and B
$UL[V]$	nilpotent part of the derivation algebra of $L[V]$
$UL[V,d]$	nilpotent part of the derivation algebra of $L[V,d]$
$\mathrm{GL}_n(R)$	group of invertable $n \times n$ matrices over a ring R
$\mathrm{SL}_n(R)$	group of $n \times n$ matrices with determinant 1 over a ring R
$\mathrm{SL}_n^{\pm 1}(R)$	group of $n \times n$ matrices with determinant ± 1 over a ring R
$\mathcal{F}_n$	free group on n generators
$\Gamma(n,d)$	free nilpotent group on n generators of nilpotency class d
$G(n,d)$	free unilpotent group on n generators of nilpotency class d
$\operatorname{Aut} \Gamma$	automorphism group of the group Γ
$\operatorname{Aut}^k \Gamma$	k-th group in the automorphism tower of the group Γ
$\operatorname{Cen}(\Gamma)$	center of the group Γ
$[A,B]$	commutator of the subgroup A and B
$M^{\mathfrak{g}}$	elemnts in the module M fixed by the algebra $\mathfrak{g}$
M^G	elemnts in the module M fixed by the group G

Introduction

1. General

A classical result of Wielandt [**24**] from the 1930-es asserts that the automorphism tower stabilizes for any finite group Γ with trivial center. The original proof was simplified by Schenkman in [**20**]. This proof can be modified to yield an effective bound on the height (see [**18**]).

There are two possible directions to generalize this result, one is to start with an infinite group Γ and the other is to start with a group with non trivial center. The straightforward generalization of Wielandt's result does not hold in any of these two cases.

If the original group Γ has trivial center the automorphism tower consist of an increasing sequence of group and this sequence can be extended to a transfinite sequence. Recently Hamkins [**9**] proved that the transfinite automorphism tower stabilizes, at some (transfinite) level, for any group Γ (even not finitely generated).

To the best of our knowledge there is no generalization of Wielandt's theorem for a large class of groups with centers. It is not known if the analog of this theorem holds for finite p-groups. One of the few cases where the generalization of Wielandt's result is known is the case of finitely generated abelian groups.

The most general version of Wielandt's theorem for infinite groups with center does not hold — there are known examples of polycyclic groups with infinite automorphism tower (see [**11**]). Knowing that the automorphism tower stabilizes for abelian groups and does not stabilize for polycyclic groups, Baumslag conjectured that for

any finitely generated nilpotent group Γ, the automorphism tower of Γ stabilizes after finitely many steps.

There has been almost no progress in proving Baumslag's conjecture during the last 50 years. In [**4**], Formanek and Dyer proved that the automorphism tower of the free nilpotent group of class 2 stabilizes at the second level, when the number of generators is different than 3. For the three generator case, the tower stabilizes at the third level. Similar results are known for other nilpotent groups, like the group of $n \times n$ upper triangular matrices, and in all these case the automorphism tower is very short.

The description of the automorphism tower is also know for the case of free groups. It was proved by Dayer and formanek (see [**5**]) in the case of finitely generated groups and by Tolstykh (see [**21**]) in the case of infinitely many generators.

Baumslag's conjecture was verified for several classes of groups and no example of a nilpotent group with infinite automorphism tower has been found. I started this project hoping to find a counter example among the free nilpotent groups. Some computations in the free Lie algebras and their derivations suggested that it may be possible to find such a counterexample among the free 2 generated nilpotent groups of odd nilpotency class.

It turns out (see Theorems 4.4.6, 4.5.5 and 4.6.5) that the free nilpotent group is too big and its automorphism tower stabilizes after finitely many steps. I still have some hope that it is possible to construct an example of a relatively free group in some nilpotent variety, whose automorphism tower does not stabilize. However it seems that in this case the tower is going to stabilize weakly, i.e., for some i the groups $\operatorname{Aut}^i \Gamma$ and $\operatorname{Aut}^{i+1} \Gamma$ will be isomorphic but the natural homomorphism between them,

sending every element to the inner automorphism obtained by conjugation with this element will not be an isomorphism.

The main result of this dissertation is the following theorem

THEOREM . *The automorphism tower of the free nilpotent group $\Gamma(n, d)$ on n generators and nilpotency class d stabilizes after finitely many steps.*

The idea of the proof of the above theorem is to embed every group which appears in the automorphism tower of Γ into a suitable Lie group over the reals and first study the automorphism group of this Lie group using the Lie theory. Using the connection between that automorphism of a Lie group and the derivations of its Lie algebra I first study the derivation tower of the free nilpotent Lie algebra. It turns out that derivation tower is very short and that it is not a good approximation of the automorphism tower of the free nilpotent group (although it is a very good approximation of the automorphism tower of the free unipotent group). In chapter 3, I define the notions of 'restricted derivation algebra' and 'tower of restricted derivations' which is a better approximation to the automorphism tower of the free nilpotent group. After obtaining a description of this tower I used rigidity results to obtain a similar description of the automorphism tower of the free nilpotent group.

2. Structure of the manuscript

This dissertation consist of 4 chapters. The first chapter describes free nilpotent Lie algebras and their derivations. The main result in this chapter is Theorem 1.4.11, which describes the generators of the derivation algebra.

In the second chapter I study the automorphism group of the free nilpotent group, using the derivation algebra of the free nilpotent Lie algebra. Theorem 2.4.1 gives a necessary and sufficient condition for this group to have Kazhdan property T.

The third chapter is devoted to the derivation tower of the free nilpotent Lie algebras. Since the derivation tower stabilizes after 3 steps and it has no connection with the automorphism tower of the free nilpotent groups, I define the notion of restricted derivations and study the tower of restricted derivations. The main results in this chapter are Theorems 3.9.1 and 3.10.1, which show that the derivation tower stabilizes and give bound of its height.

In the last chapter I study the automorphism tower of the free nilpotent group. Each group of this tower is embedded in certain Lie group (over the reals). Using rigidity results I can (in most cases) embed the next group in the gower into the automorphism group of the Lie group. This allows me to generalize the results about the stabilization of the derivation tower to Theorems 4.4.6 and 4.5.5, which give the stabilization of the derivation tower of the free nilpotent groups in different cases if the number of generators is more then 2. The case of 2 generated free nilpotent groups is a bit different because not all groups from the automorphism tower can be embedded into Lie groups. In section 4.6, I show how this problem can be avoided and Theorem 4.6.5 gives the stabilization of the automorphism tower in this case.

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CHAPTER 1

Free Nilpotent Lie Algebras

In this chapter we discuss free nilpotent Lie algebras and their derivations. The main result is Theorem 1.4.11 which describes the generating set of the derivation algebra of the free nilpotent Lie algebra if the nilpotency class is small compared to the number of generators.

1.1. Free Associative and Lie Algebras

Let V be the linear space spanned by (formal letters) $x_1, \ldots, x_n$. Let $\mathbb{R}\langle V\rangle = \mathbb{R}\langle x_1, \ldots, x_n\rangle$ be the free associative algebra over $\mathbb{R}$ generated by V, i.e., the algebra with a basis consisting of all words (including the empty one) on the letters x_i, where the product of two words is their concatenation.

This algebra is called the free associative algebra because it satisfies the following universal property:

PROPERTY 1.1.1. For any associative algebra A (over $\mathbb{R}$) and any elements $y_i \in A$ there exists a unique homomorphism $\phi : \mathbb{R}\langle x_i\rangle \to A$ of associative algebras, such that $\phi(x_i) = y_i$.

The algebra $\mathbb{R}\langle V\rangle$ has two natural gradings - the first one is a $\mathbb{Z}$-grading and the degree of each word in the basis is given by its length. We will denote the degree of a homogeneous element $a \in \mathbb{R}\langle V\rangle$ by $\deg a$. Using the grading the algebra $\mathbb{R}\langle V\rangle$ can

be written as

$$\mathbb{R}\langle V\rangle = \bigoplus_{i=0}^{\infty} \mathbb{R}\langle V\rangle^{(i)},$$

where $\mathbb{R}\langle V\rangle^{(0)} = \mathbb{R}.1$ and $\mathbb{R}\langle V\rangle^{(1)} = V$.

By the universal property of $\mathbb{R}\langle V\rangle$ it can be seen that the automorphism group $GL(V)$ of the space V acts naturally on the algebra and this action preserves the grading. Each homogeneous component $\mathbb{R}\langle V\rangle^{(i)}$ is a polynomial representation of $GL(V)$ which is isomorphic as representation to the i-th tensor power of the standard module. Therefore, it decomposes as a direct sum of irreducible representations

$$\mathbb{R}\langle V\rangle^{(i)} \simeq \bigoplus_{\lambda} m_\lambda V^\lambda,$$

where the sum is over all partitions λ of i with no more than n parts and m_λ are some positive integers. Here V^λ denotes the irreducible GL_n module corresponding to the partition λ.

The second grading of the algebra $\mathbb{R}\langle V\rangle$ is given by the multi degree — let us denote by $\deg_{x_i} \omega$ the number of occurrences of the letter x_i in the word ω. The multi degree of a basis word ω is defined as

$$\operatorname{mdeg} \omega = (\deg_{x_1} \omega, \dots, \deg_{x_n} \omega).$$

Now we can define the second grading (this grading is $\mathbb{Z}^n$ valued) of the algebra $\mathbb{R}\langle V\rangle$. Note that the action of $\mathrm{GL}(V)$ does not preserve this grading.

Let I_A be the ideal in $\mathbb{R}\langle V\rangle$ generated by the space V. This is an ideal of codimension 1 and it is called the augmentation ideal. The powers of this ideal form a filtration of the algebra, which coincide with the filtration coming from the degree grading.

DEFINITION 1.1.2. The quotient $\mathbb{R}\langle V, d\rangle = I_A/I_A^{d+1}$ is called free nilpotent algebra of class d because they satisfy a similar universal property as $\mathbb{R}\langle V\rangle$ but with respect to all nilpotent algebras of class d (note that these algebras do not have a unit). The basis of these algebras consists of all nonempty words of length less or equal to d and they can be decomposed into homogenous components as follows:

$$\mathbb{R}\langle V, d\rangle = \bigoplus_{i=1}^{d} \mathbb{R}\langle V, d\rangle^{(i)} = \bigoplus_{i=1}^{d} \mathbb{R}\langle V\rangle^{(i)}.$$

Let us consider $\mathbb{R}\langle V\rangle$ as a Lie algebra with respect to the bracket $[f, g] = fg - gf$, and let $L[V] = L[x_1, \ldots, x_n]$ denote the Lie subalgebra generated by x_i-es. The algebra $L[V]$ is called the free Lie algebra because it satisfies the universal property:

PROPERTY 1.1.3. For any Lie algebra L and any elements $y_i \in L$ there exists a unique Lie algebra homomorphism $\phi : L[x_i] \to L$ such that $\phi(x_i) = y_i$.

Since the algebra $L[V]$ is embedded into $\mathbb{R}\langle V\rangle$ it inherits from it the two gradings and the $\mathrm{GL}(V)$ module structure. With respect to the $\mathbb{Z}$ grading the algebra $L[V]$ decomposes as

$$L[V] = \bigoplus_{i=1}^{\infty} L[V]^{(i)},$$

where $L[V]^{(i)}$ is the homogenous component of degree i and we have that $L[V]^{(1)} = V$.

Let $I_L = I_A \cap L[V]$ be the augmentation ideal of the algebra $L[V]$ - this is the ideal generated by the spaces V which by definition coincide with the algebra $L[V]$. The powers of this ideal form a filtration of the algebra, which coincides with the filtration coming from the degree grading, i.e., we have

$$L[V]^{(i)} \simeq I_L^i/I_L^{i+1}.$$

As in the associative case, each homogenous component $L[V]^{(i)}$ is a polynomial representation of $\mathrm{GL}(V)$, which decomposes as a direct sum of irreducible representations

$$L[V]^{(i)} \simeq \bigoplus_{\lambda} l_\lambda V^\lambda,$$

where the sum is over all partitions λ of i with no more than n parts and l_λ are some nonnegative integers. The multiplicities l_λ are positives for all λ except the following: $\lambda = [i]$, for $i > 1$; $\lambda = [1^k]$ for $k \leq n$; and $\lambda = [2^2]$ and $\lambda = [2^3]$.

Definition 1.1.4. Finally, we can define the free nilpotent Lie algebra $L[V, d]$ of nilpotency class d as $L[V]/I_L^{d+1}$ or equivalently as the Lie subalgebra of $R\langle V, d\rangle$ generated by the space V. These algebras decompose as

$$L[V, d] = \bigoplus_{i=1}^{d} L[V, d]^{(i)} = \bigoplus_{i=1}^{d} L[V]^{(i)}.$$

1.2. Derivation of free algebras

We start with the definition of the derivation of an algebra.

Definition 1.2.1. Let A be an algebra. An endomorphism δ of A (as a linear space) is called a derivation, if for any elements $x, y \in A$ we have

$$\delta(xy) = x\delta(y) + \delta(x)y.$$

Direct computation shows that the commutator of two derivations is again a derivation, which gives that the set of all derivations forms a Lie subalgebra of $\mathrm{End}(A)$, which is denoted as $\mathrm{Der}(A)$.

If the algebra A has a grading then this grading induces a grading on the algebra $\mathrm{Der}\, A$. We say that the derivation δ is homogenous of degree k, if and only if, we have that $\deg \delta(x) = k + \deg x$ for any homogenous element x.

Also, note that if a group G acts on the algebra A by automorphisms, then this action induces an action on $\mathrm{End}(A)$. It is easy to see that $\mathrm{Der}(A)$ is a G invariant subspace, therefore we have a natural action of the group G onto the derivation algebra of A.

Finally, note that a derivation d is determined by its values on the generators of the algebra, because if we know them, using the Leibnitz rule we can reconstruct the values of d on all elements. This shows that the map $res_{\mathrm{gen}} : \mathrm{Der}\, A \to \mathrm{Hom}(\mathrm{Gen}\ A, A)$ is an injection. In general this map is not a surjection because not all maps can be extended to derivations of the whole algebra A.

There is one very important case where this map is a surjection — if the algebra A is the relatively free algebra in certain variety generated by the space V and the ground field is infinite, then the map $\mathrm{res}_{\mathrm{gen}} : \mathrm{Der}\, A \to \mathrm{Hom}(V, A)$ is a surjection. We will use this result only in the cases of free (and free nilpotent) associative and Lie algebras where it is well known.

Derivations of the free associative algebra

Let us consider the algebra $\mathbb{R}\langle V\rangle$ and its derivations — since this algebra is free in the class of all associative algebras, we have that

$$\mathrm{Der}\, \mathbb{R}\langle V\rangle \simeq \mathrm{Hom}(V, \mathbb{R}\langle V\rangle).$$

The algebra $\mathbb{R}\langle V\rangle$ has two gradings, which induces the gradings on the derivation algebra. For a derivation d we will denote by $\deg d$ and $\mathrm{mdeg}\, d$ its degree and multi degree, respectively.

The degree grading gives the following decomposition into homogenous components

$$\operatorname{Der}\mathbb{R}\langle V\rangle = \bigoplus_{i=-1}^{\infty} \operatorname{Der}\mathbb{R}\langle V\rangle^{(i)} \simeq \bigoplus_{i=-1}^{\infty} \operatorname{Hom}(V, \mathbb{R}\langle V\rangle^{(i+1)}),$$

where the isomorphism is as linear spaces. Actually this is also a $\mathrm{GL}(V)$ module isomorphism because the action of $\mathrm{GL}(V)$ preserves the grading by the degree.

Derivations of the free Lie algebra

The algebra $L[V]$ is a Lie subalgebra of $\mathbb{R}\langle V\rangle$, which generates it as an associative algebra. This implies that we can embed the derivation algebra of $L[V]$ into the derivation algebra of $\mathbb{R}\langle V\rangle$. Using the decomposition of $\operatorname{Der}\mathbb{R}\langle V\rangle$ into homogenous components we can see that

LEMMA 1.2.2. *The algebra* $\operatorname{Der} L[V]$ *has the following decomposition into homogenous components*

$$\operatorname{Der} L[V] = \bigoplus_{i=0}^{\infty} \operatorname{Der} L[V]^{(i)},$$

where $\operatorname{Der} L[V]^{(0)} = \mathfrak{gl}(V)$ *and* $\operatorname{Der} L[V]^{(i)} \simeq \operatorname{Hom}(V, L[V]^{(i+1)})$ *(as a* $\mathrm{GL}(V)$ *module). There is no homogenous component of degree* -1 *as in the case of free associative algebras, because the Lie algebra* $L[V]$ *does not have homogenous component of degree* 0.

Let us look closely at the Lie algebra structure of $\operatorname{Der} L[V]$ with respect to this decomposition.

REMARK 1.2.3. The Lie algebra structure on $\operatorname{Der} L[V]$ is given by:

The adjoint action of $\mathfrak{gl}(V)$ on $\operatorname{Der} L[V]^{(k)} = \operatorname{Hom}(V, L[V]^{(k+1)})$, coincides with the adjoint action of the Lie algebra of the group $GL(V)$ and it comes from the natural action of $\mathfrak{gl}(V)$ on V and $L[V]^{(k+1)}$;

The element $1 \in \mathfrak{gl}(V)$ acts as the degree derivation of $\operatorname{Der} L[V]$;

The Lie bracket of the elements f and g, such that $f \in \operatorname{Hom}(V, L[V]^{(k+1)})$ and $g \in \operatorname{Hom}(V, L[V]^{(l+1)})$ is defined to satisfy $D_{[f,g]} = [D_f, D_g]$. This is equivalent to

$$[f, g](x) = D_f(g(x)) - D_g(f(x)),$$

for any $x \in V$.

The action of $\mathrm{GL}(V)$ on the algebra $\operatorname{Der} L[V]$ is not polynomial (in general), because the action of $\mathrm{GL}(V)$ on the dual space V^* is not. However if we restrict the action to the subgroup $\mathrm{SL}(V)$, it becomes polynomial.

Definition 1.2.4. A poly homogeneous derivation δ of the algebra $L[V]$ is called positive $\operatorname{mdeg} \delta \geq 0$, i.e., we have that $\deg_{x_i} \delta \geq 0$ for all i .

We will call a derivation δ totaly positive if the $\mathrm{GL}(V)$ module generated by δ contains a basis of positive derivations.

The following lemma gives a more natural classification of the totally positive derivations.

Lemma 1.2.5. *The derivation δ is totally positive if and only if the action of* $\mathrm{GL}(V)$ *on the submodule generated by δ is polynomial.*

Proof. We will not prove this lemma directly here. The statement of the lemma follows easily from Theorem 1.3.8. □

Let $UL[V]$ denote the 'nilpotent' radical of $\operatorname{Der} L[V]$, i.e.,

$$UL[V] = \bigoplus_{k=1}^{\infty} \operatorname{Der} L[V]^{(k)}.$$

Derivations of the free nilpotent Lie algebras

The derivation algebra of $L[V,n]$ has a similar description:

LEMMA 1.2.6. *The derivation algebra of* $L[V,n]$ *has the following decomposition*

$$\operatorname{Der}(L[V,d]) = \bigoplus_{k=0}^{d-1} \operatorname{Der} L[V]^{(k)}.$$

Let $UL[V,d]$ denote the nilpotent radical of $\operatorname{Der}(L[V,d])$, i.e.

$$UL[V,d] = \bigoplus_{k=1}^{d-1} \operatorname{Der} L[V]^{(k)}.$$

REMARK 1.2.7. The center of the Lie algebra $UL[V,d]$ is equal to $\operatorname{Der} L[V]^{d-1}$. $UL[V,d]$ is also a nilpotent Lie algebra of class $d-1$.

PROOF. For any $f \in L[V]$ we can define the inner derivation $\operatorname{ad}(f)$ as $\operatorname{ad}(f)(g) = [f,g]$. Direct computations gives that for any derivation δ of the algebra $L[V]$ or $L[V,d]$, we have $[\delta, \operatorname{ad} x] = \operatorname{ad}(\delta(x))$. This computation shows that, if a derivation δ lies in the center of the algebra $UL[V,d]$ then we have $\delta(x) \in \operatorname{Cen}(L[V,d])$ for any $x \in L[V,d]$. By taking for x the generators of the algebra $L[V,d]$ we see that $\deg \delta = d-1$, i.e. $\operatorname{Cen} UL[V,d] \subset \operatorname{Der} L[V]^{d-1}$. The other inclusion follows from trivial degree arguments. The nilpotent class is at least $d-1$ because the image of the map ad is isomorphic to $L[v,d]/\operatorname{Cen}(L[V,d]) = L[V,d-1]$, which is a nilpotent algebra of class $d-1$. The nilpotency class is at most $d-1$ by degree arguments. $\square$

In the next chapters we will see that many properties of the automorphism tower of the free nilpotent group depend on the generating set of the algebra $UL[V,d]$.

1.3. Some computations in Lie algebra $UL[V]$

In this section we define notations for some elements in the algebra $UL[V]$ which will be used later. We also compute the Lie bracket between these elements.

DEFINITION 1.3.1. Let h be an element in $\mathbb{R}\langle V\rangle$. Let us define a derivation $\mathrm{AD}_h \in \operatorname{Der} \mathbb{R}\langle V\rangle$ by

$$\mathrm{AD}_h(x) = (h_{|x_i \to \mathrm{ad}(x_i)})(x)$$

for any $x \in V$ and extend it to the whole algebra $R\langle V\rangle$ by the Leibnitz rule. By construction we have that $\mathrm{AD}_h(x) \in L[V]$ for all $x \in V$, therefore we can consider AD_h also as a derivation of the Lie algebra $L[V]$. Note that if $h \in L[V] \subset \mathbb{R}\langle V\rangle$, then $\mathrm{AD}_h = \mathrm{ad}(h)$ is the inner derivation obtained by the element h.

Finally, we can view AD as a linear map from $\mathbb{R}\langle V\rangle$ to $UL[V] \subset \operatorname{Der} L[V]$. We will call the derivations in the image of the map AD associative derivations.

REMARK 1.3.2. The map AD is not injective. For example if $n = 2$ then

$$\mathrm{AD}_{[x_1,x_2]x_1} = 0,$$

and if $n = 3$, then

$$\mathrm{AD}_{[[x_1,x_2],[x_1,x_3]][x_1,x_2]x_1} = 0.$$

Note that if we restrict the map AD to the elements of degree at most n it becomes injective, but there are homogeneous elements of degree $n+1$ in the kernel of AD. In the case of infinitely many variables ($n = \infty$), the map AD is injective.

PROOF. Using Shirshov's basis of the free Lie algebra it can be seen that if f does not depend on x then $\mathrm{AD}_f(x) \neq 0$. This shows that the kernel of the map AD is trivial in the case of infinitely many variables and in the case of n variables there are no elements in the kernel of degree less than n. A more complicated argument based on the fact that in $L[V]$ there are no $\mathrm{GL}(V)$ modules corresponding to the partition $[1^n]$, gives that there are also no elements of degree n in $\ker \mathrm{AD}$.

We can construct an element in $\ker \mathrm{AD}$ of degree $n+1$ using the following idea — define a non trivial element $f \in L[x_i, x_{n+1}]$ by

$$f(x_i; x_{n+1}) = \sum_{\sigma \in S_{n+1}} [[x_1, x_{\sigma_1}, \dots, x_{\sigma n+1}]$$

Using Shirshov's basis we can rewrite f in such a way that all commutators start with x_{n+1}. Therefore, there exists $h \in R\langle V\rangle$ of degree $n+1$ such that $f(x_i; x_{n+1}) = \mathrm{AD}_h\, x_{n+1}$. By construction, $f(x_i; x_j) = 0$ for $j = 1, \dots, n$. Therefore, AD_h acts trivially on the space V. □

Remark 1.3.3. The map AD is surjective only if $n = 2$. For example, the derivation δ defined by $\delta(x_1) = [x_2, x_3]$ and $\delta(x_i) = 0$ for $i \neq 1$ does not lie in $\mathrm{Im}(\mathrm{AD})$.

Lemma 1.3.4. *In the Lie algebra $UL[V]$ we have the following equalities:*

(1) $[g, \mathrm{AD}_f] = \mathrm{AD}_{g(f)}$ *for any* $g \in \mathfrak{gl}$ *and* $f \in \mathbb{R}\langle V\rangle$, *i.e.* AD *is a map of* $\mathfrak{gl}$ *modules.*
(2) $[\mathrm{AD}_f, \mathrm{AD}_g] = \mathrm{AD}_{[g,f]}$ *for any* $f, g \in L[V]$.
(3) $[\delta, \mathrm{AD}_f] = \mathrm{AD}_{\delta(f)}$ *for any* $f \in L[V]$ *and* $\delta \in UL[V]$.
(4) $[\mathrm{AD}_{h_1}, \mathrm{AD}_{h_2}] = \mathrm{AD}_h$ *where* $h = -[h_1, h_2] + \mathrm{AD}_{h_1}(h_2) - \mathrm{AD}_{h_2}(h_1)$.

Proof. The proofs of all these equalities are similar, so here we only prove the third and fourth.

$$[\delta, \mathrm{AD}_f](x) = \delta(\mathrm{AD}_f(x)) - \mathrm{AD}_f(\delta(x)) = \delta([x, f]) - [\delta(x), f] =$$

$$= [\delta(x), f] + [x, \delta(f)] - [\delta(x), f] = [x, \delta(f)] = \mathrm{AD}_{\delta(f)}(x).$$

Similarly,

$$[\mathrm{AD}_{h_1}, \mathrm{AD}_{h_2}](x) = \mathrm{AD}_{h_1}(\mathrm{AD}_{h_2}(x)) - \mathrm{AD}_{h_2}(\mathrm{AD}_{h_1}(x)) =$$

$$\mathrm{AD}_{h_2 h_1}(x) + \mathrm{AD}_{\mathrm{AD}_{h_1}(h_2)}(x) - \mathrm{AD}_{h_1 h_2}(x) - \mathrm{AD}_{\mathrm{AD}_{h_2}(h_1)}(x) = \mathrm{AD}_H(x),$$

because $\mathrm{AD}_f(\mathrm{AD}_g(x)) = \mathrm{AD}_{gf}(x) + \mathrm{AD}_{\mathrm{AD}_f(g)}(x)$. □

COROLLARY 1.3.5. The space $I_L = \{\mathrm{AD}_f \,|\, f \in L[V]\}$ is an ideal in the algebra $UL[V]$ (the ideal I_L coincides with the space of all inner derivations of the algebra $L[V]$) and $I_A = \mathrm{Im}(\mathrm{AD})$ is a subalgebra. However, by (4) from the previous lemma it can be seen that it is not a Lie algebra homomorphism.

DEFINITION 1.3.6. Let us define a derivation ${}_{x_i}\mathrm{AD}_h$ by

$$ {}_{x_i}\mathrm{AD}_h(x_i) = \mathrm{AD}_h(x_i) \qquad {}_{x_i}\mathrm{AD}_h(x_j) = 0, \text{ for } i \neq j. $$

We will also need the derivation ${}_{x_i}\Delta_f$ for $f \in L[V]$ defined by

$$ {}_{x_i}\Delta_f(x_i) = f \qquad {}_{x_i}\Delta_f(x_j) = 0, \text{ for } i \neq j. $$

The next lemma gives the commutators in UL of the elements AD_f and ${}_x\Delta_f$.

LEMMA 1.3.7. *In the Lie algebra $UL[V]$ we have the following equalities:*

(1) $[\delta, {}_{x_i}\mathrm{AD}_h] = {}_{x_i}\mathrm{AD}_{\delta(h)}$ *for any* $\delta(x_i) = 0$ *and* $\deg_{x_i} \delta(x_j) = 0$.

(2) $[{}_{x_i}\mathrm{AD}_h, {}_{x_j}\mathrm{AD}_g] = {}_{x_i}\mathrm{AD}_{{}_{x_j}\mathrm{AD}_{g(h)}} - {}_{x_j}\mathrm{AD}_{{}_{x_i}\mathrm{AD}_{h(g)}}$ *if* $i \neq j$.

(3) $[{}_{x_i}\Delta_{\mathrm{ad}^l(x_k)(x_j)}, {}_{x_j}\Delta_{\mathrm{ad}^m(x_k)(x_i)}] = {}_{x_i}\mathrm{AD}_{x_k^{l+m}} - {}_{x_j}\mathrm{AD}_{x_k^{l+m}}$ *if* i, j, k *are pairwise different indexes.*

PROOF. (1) Let us see how $[\delta, {}_{x_i}\mathrm{AD}_h]$ acts on an element x_j. There are two possibilities $j = i$

$$ [\delta, {}_{x_i}\mathrm{AD}_h](x_i) = \delta({}_{x_i}\mathrm{AD}_h(x_i)) - {}_{x_i}\mathrm{AD}_h(\delta(x_i)) = \delta(h) - 0 = {}_{x_i}\mathrm{AD}_{\delta(h)}(x_i) $$

and $j \neq i$

$$ [\delta, {}_{x_i}\mathrm{AD}_h](x_j) = \delta({}_{x_i}\mathrm{AD}_h(x_j)) - {}_{x_i}\mathrm{AD}_h(\delta(x_j)) = 0 - {}_{x_i}\mathrm{AD}_h(\delta(x_j)) = 0 = {}_{x_i}\mathrm{AD}_{\delta(h)}(x_j). $$

This shows that the left and the right side act in the same way on the generators of the algebra $L[V]$, therefore they coincide as an element in $\operatorname{Der} L[V]$.

The proofs of (2) and (3) are similar. □

The next theorem gives that $UL[V]$ is generated as a linear space by the image of the map AD and by the $\mathrm{GL}(V)$ module generated by poly homogeneous derivations which are not positive.

THEOREM 1.3.8. *The algebra $UL(n,d)$ is generated as an* GL *(or $\mathfrak{gl}$) module by the image of the map* AD *and elements ${}_{x_n}\Delta_f$, where $\deg_{x_n} f = 0$.*

PROOF. Since the action of the group GL on UL is semi simple, it is enough to prove that any simple sub module is either in the image of the map AD or generated by ${}_{x_n}\Delta_f$, where $\deg_{x_n} f = 0$. Consider a new action of GL (or $\mathfrak{gl}$) on UL defined by

$$\begin{array}{ll} g\circ\delta = g(\delta)\det g & \text{for } g\in \mathrm{GL} \\ g\circ\delta = g(\delta) + \operatorname{tr} g.\delta & \text{for } g\in\mathfrak{gl}. \end{array}$$

This makes the representation polynomial and if we have a simple sub module W of UL we can find a highest weight vector δ in W (with respect to standard Cartan sub algebra).

By the choice of the element δ, there exists a nonnegative integer s such that $e_{n,n}\circ\delta = s\delta$, here $e_{i,j}$ denote the matrix which have zeros everywhere except for the i,j place where it has a one.

Now there are two possibilities — either $s=0$ or $s\geq 0$.

If $s=0$ then $e_{n,n}$ acts trivially on $\delta(x_n)$, therefore $\delta(x_n)$ is a Lie polynomial on the letters $x_1,\ldots,x_{n-1}$, i.e., $\deg_{x_n}\delta(x_n)=0$. The element δ is highest weight vector, therefore $e_{n,i}\delta = 0$ for $i\leq n$, which implies that

$$\delta(x_i) = e_{n,i}(\delta(x_n)) = 0.$$

The last equation is the same as $\delta = {}_{x_n}\Delta_{\delta(x_n)}$.

If $s > 0$, then the standard action of GL on W is also polynomial. We will use induction on the maximal l such that $\delta(x_{n-k}) = 0$ for all $k \geq l$, to prove that δ lies in the image of the map AD. The base case is trivial since if $l = 0$ then $\delta = 0$.

Let us linearize $\delta(x_k)$ with respect to the variable x_k, i.e., let us substitute $x_k + y$ in the place of x_k in $\delta(x_k)$ and expand the result as a function on the variable y:

$$\delta(x_k)|_{x_k \to x_k+y} = \delta(x_k) + \sum F(\delta)_k^i(x_1, \ldots, x_n; y),$$

where F_k^i is a Lie polynomial with degree i with respect to y. It follows from the definition of F_k^1 that

$$F(\delta)_k^1(x_1, \ldots, x_n; x_i) = e_{k,i}(\delta(x_k)),$$

and that F_k^1 lies in the image of the map AD. Now let us define the derivation F by

$$F(\delta)(y) = \sum_k F(\delta)_k^1(x_1, \ldots, x_n; y), \text{ for } y \in V.$$

First, we want to show that the element F is also a highest weight vector for some $\mathfrak{gl}$ module isomorphic to W, which follows from the next lemma.

LEMMA 1.3.9. *Let the element $F(\delta)$ be defined in the way above. Then, for $g \in \mathfrak{gl}$ we have $gF(\delta) = F(g\delta)$.*

PROOF. In order to prove the lemma it is enough to show it for $g = e_{p,q}$. Let us compute how $e_{p,q}$ acts on $F(\delta)(x_i)$.

$$\begin{aligned}
e_{p,q}(F(\delta)(x_i)) &= \textstyle\sum_k e_{p,q}(e_{k,i}(\delta(x_k))) = \\
&= \textstyle\sum_k [e_{p,q}e_{k,i}](\delta(x_k)) + \sum_k e_{k,i}(e_{p,q}(\delta(x_k))) = \\
&= \delta_{p,i} \textstyle\sum_k e_{k,q}(\delta(x_k)) - e_{p,i}\delta(x_q) + \\
&\qquad \textstyle\sum_k e_{k,i}((e_{p,q}\delta)(x_k)) + e_{p,i}\delta(x_q) = \\
&= F(\delta)(e_{p,q}x_i) + F(e_{p,q}\delta),
\end{aligned}$$

where $\delta_{p,i}$ is the Kronecker delta. Therefore, $e_{p,q}(F(\delta)) = F(e_{p,q}\delta)$, which proves the lemma. $\square$

Let us compute $F(x_i)$

$$F(x_i) = \sum e_{k,i}\delta(x_k) = \sum_{k\leq i} e_{k,i}\delta(x_k) + \sum_{k>i} e_{k,i}\delta(x_k) = \sum_{k\leq i} e_{k,i}\delta(x_k) + (n-i)\delta(x_i),$$

i.e., $F(\delta)(x_{n-k}) = 0$ for $k \geq l$ and

$$F(\delta)(x_m) = ((n-m)\delta + e_{m,m}\delta)(x_m) = s_l\delta(x_m)$$

for $m = n-l-1$ and some positive integer $s_l \geq s > 0$. Therefore, we may use the induction assumption of the derivation $\delta - s_l^{-1}F$, i.e., we may assume that it lies in the image of AD. The derivation $s_l^{-1}F$ also lies in the image of AD, therefore the same is true for δ. This finishes the proof of the induction step and completes the proof of the theorem.

$\square$

1.4. Generators of the algebra $UL[V]$

The $\mathfrak{sl}$-invariant component of the set of generators of the algebra $UL[V]$ has direct connections with Kazhdan property T of the automorphism group of the free nilpotent group. In this section our aim is to prove Theorem 1.4.11, which fully describes the generators of $UL[V,d]$ if $d \leq n(n-1)$.

LEMMA 1.4.1. *The algebra* $\operatorname{Der} L[V]$ *is generated as a Lie algebra by* $\mathfrak{gl}$ *and the elements which generate* $UL[V]/\big[UL[V], UL[V]\big]$ *as* $\mathfrak{gl}$ *module.*

PROOF. For any d, the algebra $UL[V,d]$ is a nilpotent algebra. Therefore, it is generated as a Lie algebra by the elements in $UL[V,d]/\big[UL[V,d], UL[V,d]\big]$. Using limit arguments and the fact that

$$\bigcap_d \ker(UL[V] \to UL[V,d]) = 0$$

we see that $UL[V]$ behaves like a nilpotent Lie algebra. Therefore, it is generated by the elements which generate $UL[V]/\big[UL[V], UL[V]\big]$ as $\mathfrak{gl}$ module. $\square$

The algebra $\mathfrak{sl}$ is simple. Therefore, the $\mathfrak{sl}$ module $\big[UL[V], UL[V]\big]$ has a (non invariant) complement $U_{\text{gen}}[V]$, which is also a graded $\mathfrak{sl}$ module. Thus, we have the isomorphism

$$UL[V]/\big[UL[V], UL[V]\big] \simeq U_{\text{gen}}[V].$$

Remark 1.4.2. This module is not uniquely defined in general but its isomorphism type does not depend on the choice of U_{gen} (if d is small compared to n it follows from Theorem 1.4.11 that the homogenous components of $U_{\text{gen}}[V]$ of degree less than d are uniquely defined). In this section we want to treat the elements of U_{gen} as elements in UL, that is why we define U_{gen} as a submodule. In general it is better to use the above isomorphism to define U_{gen} as an abstract $\mathfrak{gl}$ module (see Chapter 3).

Similarly, we can define $U_{\text{gen}}[V, d]$, which satisfies

$$U_{\text{gen}}[V, d] = U_{\text{gen}}[V] \cap UL[V, d].$$

Lemma 1.4.3. *Let δ be a derivation such that $\deg_{x_i} \delta = -1$ for some index i and $\deg \delta > 1$. Then δ lies in the commutator subalgebra of $UL[V]$.*

Proof. If $\deg_{x_i} \delta = -1$, then $\delta = {}_{x_i}\Delta_f$, for some f with $\deg_{x_i} f = 0$. Using Lemma 1.3.7 we can see that

$$\big[{}_{y_i}\Delta_h,\ {}_x\Delta_{[y_i, y_j]}\big] = {}_x\Delta_{[h, y_j]},$$

for any $i \neq j$ and $h \in L[V]$ such that $\deg h > 1$, but $\deg_x h = 0$. Any element $f \in L[V]$, which does not depend on x, can be expressed as a linear combination of elements $[h, y_j]$. This, together with the observation above, proves the lemma. This

proof does nor work when $n = 2$ (we can not find two different y-es), but in that case there are no derivations δ such that $\deg_{x_i} \delta = -1$, since the map AD is surjective. $\square$

This result together with lemma 1.3.8 gives that the module U_{gen} 'almost' lies in the image of the map AD:

THEOREM 1.4.4. *We have that*

$$UL[V]^{(1)} \subset U_{\text{gen}}[V] \subset \mathrm{Im}(\mathrm{AD}) + UL[V]^{(1)}.$$

REMARK 1.4.5. The elements $\mathrm{AD}_{x_i^k}$ for $k \geq 1$ do not lie in $\big[UL[V], UL[V]\big]$, i.e., they lie in $U_{\text{gen}}[V]$.

PROOF. Using multi degree arguments we can see that the only way to express a derivation of multi degree $(k, 0, \dots, 0)$ as a commutator of two derivations is if they have multi degrees $(l, 0, \dots, 0)$ and $(k - l, 0, \dots, 0)$ or $(l, 1, -1, 0, \dots, 0)$ and $(k - l, -1, 1, 0, \dots, 0)$. Using the computations from Lemmas 1.3.4 and 1.3.7, we can see that in both cases the commutator of these derivations has trivial projection in the submodule generated by $\mathrm{AD}_{x_i^k}$. $\square$

In order to show that these elements generate U_{gen} for small values of d, we need to prove several lemmas which allow us to 'simplify' elements in UL modulo $\big[UL[V], UL[V]\big]$.

From now on, we need to distinguish one element of the set $X = \{x_1, \dots, x_n\}$ from the others. Therefore, we will split X as $X = \{x\} \cup Y$, where $Y = \{y_1, \dots, y_l\}$, $l = n - 1$, and $x \notin Y$.

First we will prove a lemma about free associative algebras, which will be used later in lemma 1.4.7 to simplify the elements in UL modulo the commutator subalgebra.

LEMMA 1.4.6. *Let $f \in \mathbb{R}\langle x, Y\rangle$ be an element which lies in the commutator ideal of $\mathbb{R}\langle x, Y\rangle$ (i.e., f has a trivial projection in the polynomial ring $\mathbb{R}[x, Y]$). Then f can be expressed as a linear combination of elements of the type ${}_x\Delta_g\, h$, where $h \in \mathbb{R}\langle x, Y\rangle$ and ${}_x\Delta_g$ is a derivation of $\mathbb{R}\langle x, Y\rangle$, which kills all y_i-es and sends x to a homogeneous element g in the free Lie algebra $L[x, Y]$ of degree at least 2.*

PROOF. The free associative algebra $\mathbb{R}\langle x, Y\rangle$ is the universal enveloping algebra of the free Lie algebra $L[x, Y]$. Therefore, any element f in $\mathbb{R}\langle x, Y\rangle$ can be expressed as a linear combination of the elements

$$\mathrm{Sym}_k(h_1, \ldots, h_k) = \sum_{\sigma \in S_k} h_{\sigma(1)} \ldots h_{\sigma(k)},$$

where the elements $h_i \in L[x, Y]$ are homogeneous, and $k \leq p = \deg f$.

The condition that f has a trivial projection in the polynomial algebra $\mathbb{R}[x, Y]$, implies that in the above decomposition there are no elements with $k = p$. This gives that at least one of the elements h_i has degree at least 2. Therefore, it suffices to prove the lemma only for $f = \mathrm{Sym}_k(h_1, \ldots, h_k)$. Without loss of generality, we may assume that if $\deg h_i = 1$, then either $h_i = x$ or $h_i \in Y$.

We will prove the lemma for these f's (assuming that k is fixed) using induction on the number q of elements h_i of degree greater then 1. The base case $q = 0$ is impossible, because $k < p$.

Let $q > 0$. We can permute the elements h_i, in order to have

$$\deg h_1 \geq \cdots \geq \deg h_q > 1$$

and $h_{q+1} = \cdots = h_{q+t} = x$ and the remaining h's lie in Y. In this case we have the following equality:

$${}_x\Delta_{h_1}(\mathrm{Sym}_p(x, h_2, \ldots, h_p)) =$$

$$= (t+1)\,\mathrm{Sym}_p(h_1,\ldots,h_p) + \sum_{i=2}^{q} \mathrm{Sym}_p(x, h_2, \ldots, \,{}_x\Delta_{h_1}(h_i), \ldots h_p).$$

In each term of the sum we have that the number of Lie elements of degree at least 2 is $q-1$. Then, by the induction hypothesis, $\mathrm{Sym}_k(h_1,\ldots,h_k)$ is expressible by a linear combination of the elements $D_g(h)$. This, together with the fact that the left hand side is of the same type, proves the induction step. $\square$

LEMMA 1.4.7. *Let $f \in L[x,Y]$ be an element such that $1 \le \deg_x f \le l = |Y|$. Then the derivation ${}_x\Delta_f$ is equivalent modulo $\big[UL[x,Y], UL[x,Y]\big]$ to a derivation lying in the $\mathfrak{sl}$ module generated by $\Delta_{s;n_1,\ldots,n_s}$, for $s \le l$. Here we have denoted*

$$\Delta_{s;n_1,\ldots,n_s} = {}_x\mathrm{AD}_{y_1^{n_1}xy_2^{n_2}x\ldots xy_s^{n_s}}\,.$$

PROOF. Any such element f can be expressed as a linear combination of elements

$$\big(h_{1|y_i\to\mathrm{ad}(y_i)} \circ \mathrm{ad}(x) \circ h_2 \circ \mathrm{ad}(x) \circ \cdots \circ h_s\big)\,(x),$$

where $s = \deg_x f$ and h_i are associative polynomials on y_i-es, and we have substituted y_i with $\mathrm{ad}(y_i)$ in each of them. Denote the derivation which kills all y_i-es and sends x to the element above by ${}_x\Delta_{s;h_1,\ldots,h_s}$.

Using Lemma 1.3.4 part 5, we can show that

$$\sum_{j=1}^{s} {}_x\Delta_{s;h_1,\ldots,\nu(h_j),\ldots,h_s} \in \big[UL[x,Y], UL[x,Y]\big],$$

for any derivation ν such that $\nu(x)=0$ and $\nu(y_i) \in L[Y]$.

Let us fix $s \le l$. We will show that the derivation ${}_x\Delta_{s;h_1,\ldots,h_s}$ is equivalent to a derivation which lies in the $\mathfrak{sl}$ module generated by $\Delta_{s;n_1,\ldots,n_s}$. To show this we will use induction on $(\deg h_1,\ldots,\deg h_s)$ with respect to lexicographical order.

Note that the base case is trivial because ${}_x\Delta_{s;1,\ldots,1} = 0$. On each step we will use induction on $1 \le k \le s$ to show that such ${}_x\Delta_{s;h_1,\ldots,h_s}$ lies in $\mathfrak{sl}$ the module generated

by

$$ {}_x\Delta_{s;y_1^{n_1},\dots,y_k^{n_k},h_{k+1},\dots,h_s}. $$

Thus, the second induction allows us to make the induction step for the first one.

Therefore, the only thing left is to show the induction step for the second induction (the second induction also has a trivial base case).

Let us first consider the case where h_{k+1} lies in the commutator algebra of $\mathbb{R}\langle Y\rangle$. After rearranging the variables as follows: $x := y_{k+1}$; $y_i := y_i$ for $i \le k$, and $y_i := y_{i+1}$ for $i > k$ we can apply Lemma 1.4.6 and, without loss of generality, we can assume that $h_{k+1} = D_{|y_{k+1}\to g}t$ for some Lie polynomial g and an associative polynomial t. Using Lemma 1.3.4 we notice that

$$ [D_{|y_{k+1}\to g}, \Delta_{s;y_1^{n_1},\dots,y_k^{n_k},t,h_{k+2},\dots,h_s}](x) = $$

$$ = \Delta_{s;y_1^{n_1},\dots,y_k^{n_k},h_{k+1},\dots,h_s} + \sum_{j=k+2}^{s} \Delta_{s;y_1^{n_1},\dots,y_k^{n_k},t,h_{k+2},\dots,D_{|y_{k+1}\to g}h_j,\dots,h_s}. $$

All terms in the sum have multi-degree less than $(n_1,\dots,n_s)$, because at position $k+1$ the degree is

$$ \deg t = \deg h_{k+1} + 1 - \deg g < \deg h_{k+1} = n_{k+1}, $$

and for all $j \le k$ the degree is n_j. Therefore, by the induction hypothesis they are equivalent to elements in the $\mathfrak{sl}$ module generated by $\Delta_{s;n_1,\dots,n_s}$. Therefore, the same is true for ${}_x\Delta_{s;y_1^{n_1},\dots,y_k^{n_k},h_{k+1},\dots,h_s}$.

Using this argument, we can view h_{k+1} as an element (up to equivalence) in the polynomial algebra $\mathbb{R}[Y]$. Now let us consider the Lie algebra

$$ \mathfrak{u} = \{t \in \mathfrak{sl}(Y) \mid t(y_i) = 0, \text{ for } j \ne k+1\} $$

and its action on the set of derivations of the form ${}_x\Delta_{s;y_1^{n_1},\dots,y_k^{n_k},h_{k+1},\dots,h_s}$. This set is closed under the action of the algebra $\mathfrak{u}$, because all elements in the algebra $\mathfrak{u}$ kill

the letters y_i, for $i \leq k$, and x. The elements t^l_{k+1} generate the $\mathbb{R}[Y]$ as a $\mathfrak{u}$ module. Therefore, any derivation $\delta = {}_x\Delta_{s;y_1^{n_1},\dots,y_k^{n_k},h_{k+1},\dots,h_s}$ up to equivalence lies in the $\mathfrak{sl}$ module generated by the elements ${}_x\Delta_{s;y_1^{n_1},\dots,y_{k+1}^{n_{k+1}},h_{k+2},\dots,h_s}$. This argument completes the induction step.

Thus, we have shown that up to an element in $[UL, UL]$, every derivation ${}_x\Delta_f$ with $\deg_x f \leq l$ lies in the $\mathfrak{sl}$ module generated by $\Delta_{s;n_1,\dots,n_s}$. $\square$

LEMMA 1.4.8. *If $s \geq 2$ then the derivation $\Delta_{s;n_1,\dots,n_s}$ lies in the $\mathfrak{sl}$ module generated by $\Delta_{1;y_1^n}$, for $n = \sum n_i + s - 1$, plus the commutator subalgebra of UL.*

PROOF. Let us compute the commutator of derivations $\Delta_{s-1;y_1^{n_1},\dots,y_{s-2}^{n_{s-2}},y_{s-1}^{n_s}y_s}$ and ${}_{y_s}\Delta_{\mathrm{ad}^{n_s}(y_s)(x)}$.

$$[\Delta_{s-1;y_1^{n_1},\dots,y_{s-2}^{n_{s-2}},y_{s-1}^{n_s}y_s},\ {}_{y_s}\Delta_{\mathrm{ad}^{n_s}(y_s)(x)}](x) =$$
$$= -\ {}_{y_s}\Delta_{\mathrm{ad}^{n_s}(y_s)(x)}\left(\mathrm{ad}^{n_1}(y_1)\,\mathrm{ad}(x)\dots\mathrm{ad}(x)\,\mathrm{ad}^{n_{s-1}}(y_{s-1})[x, y_s]\right) =$$
$$= \left(\mathrm{ad}^{n_1}(y_1)\,\mathrm{ad}(x)\dots\mathrm{ad}(x)\,\mathrm{ad}^{n_{s-1}}(y_{s-1})\,\mathrm{ad}(x)\,\mathrm{ad}^{n_s}(y_s)(x)\right)$$

and

$$[\Delta_{\dots},\ {}_{y_s}\Delta_{\mathrm{ad}^{n_s}(y_s)(x)}](y_s) = \Delta_{\dots}\left(\mathrm{ad}^{n_s}(y_s)(x)\right) =$$
$$= \left(\mathrm{ad}^{n_s}(y_s)\,\mathrm{ad}^{n_1}(y_1)\,\mathrm{ad}(x)\dots\mathrm{ad}(x)\,\mathrm{ad}(y_{s-1})^{n_{s-1}}\,\mathrm{ad}(x)\right)(y_s).$$

Using a permutation (the one which interchanges x and y_s) of variables we can see that the last expression is equivalent to one in the $\mathfrak{sl}$ module generated by

$$\Delta_{n_s+1;1,\dots,1,y_s y_1^{n_1} y_s \dots y_s y_{s-1}^{n_{s-1}} y_s}.$$

Using the same argument as in the previous lemma we can show that the last derivation lies in the $\mathfrak{sl}$ module generated by $\Delta_{n_s+1;1,\dots,1,y_1^{n'}}$, where $n' = \sum_{i=1}^{s-1} n_i + s - 1$.

Now using the commutator of derivations ${}_x\Delta_{\mathrm{ad}^{n_s}(x)(y_2)}$ and ${}_{y_2}\Delta_{\mathrm{ad}^{n'}(y_1)(x)}$, we can show that $\Delta_{n_s+1;1,\dots,1,y_1^{n'}} \sim {}_{y_2}\Delta_{\mathrm{ad}^{n'}(y_1)\,\mathrm{ad}^{n_s}(x)(y_2)}$.

By the argument from Lemma 1.4.7, we can see that the last derivation is equivalent to a derivation in the $\mathfrak{sl}$-module generated by $\Delta_{1;y_1^n}$ for $n = n' + n_s = \sum n_i + s - 1$. $\square$

LEMMA 1.4.9. *Let δ be a derivation such that $\deg_{y_i}\delta \geq 0$ for all i and $\sum \deg_{y_i}\delta > 1$. Then δ is equivalent (modulo the commutator subalgebra of UL) to a derivation which acts nontrivially only on the variable x.*

PROOF. Let us first consider the case when $\delta = {}_{y_i}\Delta_{[f,y_j]}$ for some indices i, j and some Lie polynomial $f \in L[x, Y]$. Using the derivations ${}_x\Delta_f$ and ${}_{y_i}\Delta_{[x,y_j]}$, it can be seen that δ is equivalent to ${}_x\Delta_{{}_{y_i}\Delta_{[x,y_j]}(f)}$, which is a derivation acting nontrivially only on x.

Now, let us consider the case when δ is such that $\delta = {}_{y_i}\Delta_{\mathrm{ad}^n(x)[f,y_j]}$ for some i, j and some Lie polynomial $f \in L[x, Y]$.

This time, using derivations ${}_{y_i}\Delta_{\mathrm{ad}^n(x)(y_i)}$ and ${}_{y_i}\Delta_{[f,y_j]}$, we can show that

$$\delta \sim {}_{y_i}\Delta_{[{}_{y_i}\Delta_{\mathrm{ad}^n(x)(y_i)}(f),y_j]}.$$

But the last derivation is of the type considered before. Therefore, it is equivalent to one acting nontrivially only on x.

Since every derivation satisfying the conditions of the lemma can be expressed as a linear combination of the derivations considered above, the lemma is proved. $\square$

LEMMA 1.4.10. *Let δ be a poly-homogeneous derivation in $UL[x, Y]$, and $\deg_x \delta < l = |Y|$. Then, if $\deg \delta > 1$, then δ is equivalent modulo $[UL, UL]$ to a derivation in the $\mathfrak{sl}$ module generated by AD_{x^n}, where $\deg \delta = n$.*

PROOF. Suppose that there exists i such that $\deg_{y_i}\delta = -1$. Then by lemma 1.4.3 δ lies in the commutator subalgebra of UL.

If we have that $\deg_{y_i} \delta > 0$ for all i, from Lemma 1.4.9 it follows easily that $\delta \sim \delta'$, where $\delta' = {}_x\Delta_f$ and $\deg_x f \leq l$. Now by Lemma 1.4.7, δ' is equivalent to a derivation in a $\mathfrak{sl}$ module generated by ${}_x\mathrm{AD}_{y_1^n}$. But from Lemma 1.3.4 we have that ${}_{y_i}\mathrm{AD}_{x^n} \sim {}_{y_j}\mathrm{AD}_{x^n}$, i.e. $l. {}_x\mathrm{AD}_{y_1^n} \sim \mathrm{AD}_{y_1^n}$, which completes the proof. $\square$

THEOREM 1.4.11. *If $d \leq n(n-1)$, then the Lie algebra $UL(n,d)$ is generated as a Lie algebra and $\mathfrak{sl}$ module by the elements: $\mathrm{AD}_{x_1^k}$, for $1 \leq k < d$, and if $n \geq 3$ the element ${}_{x_1}\Delta_{[x_2,x_3]}$.*

PROOF. If $\delta \in UL(n,d)$ is a poly-homogeneous derivation, then there exists an i such that $\deg_{x_i} \delta < n-1$. If $\deg \delta > 1$, we can apply Lemma 1.4.3 or Lemma 1.4.10, to show that

$$\delta \in \operatorname{span}_{\mathfrak{sl}}\{\mathrm{AD}_{x_1^n}\} + \big[UL[X], UL[X]\big].$$

Using this and the fact that $\operatorname{Der} L[X]^{(1)}$ is generated as $\mathfrak{sl}$ module by AD_{x_1} and ${}_{x_1}\Delta_{[x_2,x_3]}$, we can easily finish the proof. $\square$

COROLLARY 1.4.12. The same argument shows that $UL[x_1,\dots,x_n]$ is generated (as a Lie algebra and $\mathfrak{sl}$ module) by $\mathrm{AD}_{x_1^k}$, for $k \geq 1$, ${}_{x_1}\Delta_{[x_2,x_3]}$ and elements AD_h, where $h \in \mathbb{R}\langle X\rangle$ such that h is a generator of a $\mathfrak{gl}$ module corresponding to a partition λ with $\lambda_n \geq n-1$.

PROOF. Let $h \in \mathbb{R}\langle X\rangle$ be an associative polynomial, which lies in the $\mathfrak{sl}$-module M corresponding to the partition $\mu = (\mu_1,\dots,\mu_n)$, with $\mu_n < n-1$. Then, there is a generator $\tilde{h}$ of the module M such that $\deg_{x_1} \tilde{h} = \mu_n < n-1$. By Lemma 1.4.10 we have that $\tilde{h} \in \big[UL[X], UL[X]\big]$. $\square$

1.5. Conjecture

We will see that the degree of the first $\mathfrak{sl}_n$ invariant generator of $UL[V]$ determines when the automorphism group of the free nilpotent group on n generators and nilpotency class d has Kazhdan property T, and it also determines the behavior of the automorphism tower of that group. Therefore, it is useful to have a notation of that degree

DEFINITION 1.5.1. Let $h(n)$ be the minimal degree of an $\mathfrak{sl}$-invariant generator of $UL[x_1, \ldots, x_n]$, i.e.,

$$h(n) = \min\{\deg f \mid f \in U_{\text{gen}} \text{ and } [f, \mathfrak{sl}] = 0\}.$$

Theorem 1.4.11 states that $h(n) \geq n(n-1)$. We were unable to show any $\mathfrak{sl}$ invariant element which does not lie in $[UL, UL]$, unless $n = 2$. Therefore we want to make a conjecture that Theorem 1.4.11 can be significantly improved.

CONJECTURE 1.5.2. If $n \geq 3$, then the Lie algebra $UL[x_1, \ldots, x_n]$ is generated as a Lie algebra and $\mathfrak{sl}$ module by: $\mathrm{AD}_{x_1^n}$ for $n \geq 1$ and ${}_{x_1}\Delta_{[x_2,x_3]}$, i.e., $h(n) = \infty$.

If this conjecture is true then the automorphism tower of the free nilpotent groups on n generators will have relatively simple description and it will stabilizes after at most 3 steps.

REMARK 1.5.3. Let V be an infinite dimensional linear space with basis x_i, for $i = 1, \ldots$. We can define the free Lie algebra $L[V]$ and its derivation algebra in the same way as in sections 2.1 and 2.2. Then, the proof of Theorem 1.4.11 with slight modifications gives that $UL[V]$ is generated as a Lie algebra (allowing certain infinite

sums) and $\mathrm{Aut}(V)$ module by the elements $\mathrm{AD}_{x_1^k}$, for $k = 1, \ldots, \infty$, and the element ${}_{x_1}\Delta_{[x_2,x_3]}$, i.e., the conjecture 1.5.2 is true in the case $n = \infty$.

Remark 1.5.4. The analog of the above conjecture for $n = 2$ does not hold. For example, the element $\mathrm{AD}_{[x_1,x_2]^2}$ does not lie in the commutator subalgebra of $UL[x_1, x_2]$. This is true because $\mathrm{AD}_{[x_1,x_2]^2} \notin I_L$ and there is only one $\mathfrak{sl}_2$ module of degree less then 3 which is not in I_L, namely the module U generated by $\mathrm{AD}_{x_1^2}$, and direct computation shows that there are no trivial $\mathfrak{sl}_2$ submodules in $[U, U]$. This argument gives that $h(2) = 4$ and suggests that it is possible to find a counter example to Baumslag's conjecture among the 2 generated relatively free groups in some nilpotent varieties.

We tried to describe the generators of the algebra $UL(n, d)$ for some small values of n and d hoping to disprove Conjecture 1.5.2. Some computer simulations verified this conjecture for $n = 3$ and $d \leq 13$ and for $n = 4$ and $d \leq 13$. Unfortunately, we were unable to examine the case $n = 5$ because by Theorem 1.4.11, we need to consider $d > 20$ and in this case the dimension of UL is too big and the brute force approach does not work, because we do not have sufficient computer resources.

CHAPTER 2

Automorphism Group of Free Nilpotent Group

In this chapter we study the automorphism groups of free nilpotent groups. Our main goal is to prove Theorem 2.4.1 which gives partial answer to a question posed by A. Lubotzky and I. Pak in [**15**]. In the last section of this chapter we state a recent result by E. Formanek (see [**6**]) describing the center of this automorphism group. This result is later used in Chapter 4, to obtain description of the automorphism tower of free nilpotent groups.

Our main method in studying $\operatorname{Aut}\Gamma$ is to embed $\operatorname{Aut}\Gamma$ in some Lie group G over $\mathbb{R}$. Using this embedding we transfer many questions concerning the group $\operatorname{Aut}\Gamma$ to similar question about the group G. After that we use methods from Lie theory to transfer these problems to the Lie algebra of this group, which we studied in Chapter 1.

2.1. Free Nilpotent Groups

In this section we describe the free nilpotent group $\Gamma(n,d)$ of nilpotency class d on n generators. We also describe the free unipotent group $G(n,d)$, which naturally contains $\Gamma(n,d)$ as a lattice.

Definition 2.1.1. Let $\mathcal{F}_n$ be the free group on n generators which we will label $g_1,\ldots,g_n$. The elements in this group are all words on g_i and g_i^{-1} without cancellations. This group is called a free group because it satisfy the universal property:

PROPERTY 2.1.2. For any group G and any elements $a_i \in G$ there exist a unique group homomorphism $\phi : F_n \to G$, such that $\phi(g_i) = a_i$.

DEFINITION 2.1.3. Let G be a group. We can define the sequence of normal subgroups $G^{(i)}$ of G called the lower central series of the group G by $G^{(0)} = G$ and $G^{(i+1)} = [G, G^{(i)}]$, where $[A, B]$ denotes the subgroup generated by all commutators $[a, b] = aba^{-1}b^{-1}$, for $a \in A$ and $b \in B$. A group G is called nilpotent of class d if and only if $G^{(d+1)} = 1$.

Taking a quotient of the free group by a group in the lower central series we obtain a free nilpotent group.

DEFINITION 2.1.4. Let $\Gamma(n, d)$ be the free nilpotent group on n generators of nilpotency class d, i.e., $\Gamma(n, d) = \mathcal{F}_n/\mathcal{F}_n^{(d+1)}$, where $\mathcal{F}_n$ is the free group on n generators and $\mathcal{F}_n^{(d+1)}$ is the $d+1$ term of its lower center series. This group is called a free nilpotent group because it satisfies the universal property:

PROPERTY 2.1.5. For any nilpotent of class d group G and any elements $a_i \in G$ there exists a unique group homomorphism $\phi : \Gamma(n, d) \to G$, such that $\phi(g_i) = a_i$.

We want to describe the automorphism group $\operatorname{Aut}\Gamma(n, d)$. One way of doing this is to embed the group $\operatorname{Aut}\Gamma(n, d)$ as a lattice into a Lie group (over the reals) $G(n, d)$. Then, using rigidity results, the group $\operatorname{Aut}\Gamma(n, d)$ can be embedded into $\operatorname{Aut} G(n, d)$. Finally, we use the properties of the group $\operatorname{Aut} G(n, d)$ to obtain results about $\operatorname{Aut}\Gamma(n, d)$.

Our first step is to define a unipotent Lie group $G(n, d)$ such that $\Gamma(n, d)$ is a lattice in it. The general construction of embedding a discrete nilpotent group into

the unipotent group was first done by P. Hall (see [**8**]). This embedding is known as Malcev embedding — actually the Malcev embedding gives a unipotent group over $\mathbb{Q}$. In order to obtain an unipotent group over $\mathbb{R}$ we need to take its completion. Many results in this chapter as well as in Chapter 4 can be obtained by working with Lie groups over $\mathbb{Q}$ instead of Lie groups over $\mathbb{R}$. We feel that working with Lie group over the real numbers is more natural, and this is the main reason why we use Lie group over $\mathbb{R}$ in these chapters.

Definition 2.1.6. Let $G(n,d)$ be the free unipotent group over $\mathbb{R}$ on n generators $g_1,\dots,g_n$ and of class d. One way to construct the group $G(n,d)$ is to use the fact that it can be embedded in the multiplicative group of the algebra $\mathbb{R}.1+\mathbb{R}\langle V,d\rangle$. Its image is the group generated by $g_i = 1+x_i$ and operation $g \to g^\alpha$ for $\alpha \in \mathbb{R}$, which is defined formally by $(1+h)^\alpha = \sum_{k=0}^{d} \binom{k}{n} h^k$ for any $h \in I$ Here I is the augmentation ideal in $\mathbb{R}\langle x_1,\dots,x_n; d\rangle$. Using this embedding we will often assume that $G(n,d)$ is a subgroup of the multiplicative group of the algebra $\mathbb{R}.1+\mathbb{R}\langle V,d\rangle$.

The following result is well known.

Lemma 2.1.7. *The group $G(n,d)$ is a Lie group and its Lie algebra is isomorphic to the free nilpotent Lie algebra on n generators and class d. Also, the group $\Gamma(n,d)$ is a Zariski dense lattice in it.*

Proof. Using the universal properties of the groups $\Gamma(n,d)$ and $G(n,d)$ it can be seen that the map $\rho : \Gamma(n,d) \to G(n,d)$ sending g_i to $1+x_i$ is an injection, i.e., the group $\Gamma(n,d)$ can be embedded into $G(n,d)$. The image of this embedding is Zariski dense because the quotient $G(n,d)/\Gamma(n,d)$ is compact. □

Using this lemma from now on we will assume that $\Gamma(n,d)$ is a subgroup of $G(n,d)$.

2.2. Automorphism group

Now we are ready to define the automorphism group of Γ, which is the main object in this chapter.

DEFINITION 2.2.1. Let G be a group, by $\mathrm{End}(G)$ we will denote the set of all endomorphisms of the group G. Let us note that every endomorphism is uniquely determined by its values on the generators of the group G, i.e., the restriction map $\mathrm{res}_{\mathrm{gen}} : \mathrm{End}\, G \to \mathrm{Hom}(\mathrm{Gen}\, G, G)$ is an injection (here Hom denotes just the set of maps from gen$G$ to G). As in the case of algebras this map in not a surjection in general but is a surjection if and only if G is the relatively free group in some variety.

Let also denote by $\mathrm{Aut}\, G \subset \mathrm{End}(G)$ the subset of all invertible endomorphisms of the group G. By definition $\mathrm{Aut}\, G$ is a group, which is called the automorphism group of the group G. It contains a normal subgroup consisting of all inner automorphisms $\mathrm{Inn}(G) = \{\mathrm{ad}\, g | g \in G\} < \mathrm{Aut}(G)$, where $\mathrm{ad}\, g$ denotes the conjugation by the element g, i.e., $(\mathrm{ad}\, g)(h) = ghg^{-1}$.

Using the universal property of the group $G(n, d)$ it can be seen that every map from the generating set of G to G itself, can be extended to an endomorphism of G, i.e., $\mathrm{End}\, G(n, d)$ is isomorphic to $G(n, d)^{\times n}$ as a set. The automorphism group contains all the invertible endomorphisms — these are the endomorphisms which act on the abelinization as invertible linear transformations.

LEMMA 2.2.2. *The automorphism group of $G(n, d)$ satisfies the exact sequence*

$$1 \to UG(n, d) \to \mathrm{Aut}\, G(n, d) \to \mathrm{GL}_n(\mathbb{R}) \to 1$$

where $UG(n,d)$ is a unipotent group of class $d-1$ consisting of all automorphisms which act trivially on the abelinization $G/[G,G] \simeq \mathbb{R}^n$. The group $UG(n,d)$ is isomorphic to $\mathrm{Hom}(G/[G,G],[G,G])$ *as a topological space.*

In fact the exact sequence splits and we have the isomorphism

$$\mathrm{Aut}\, G(n,d) = \mathrm{GL}_n(\mathbb{R}) \ltimes UG(n,d).$$

PROOF. The group UG is unipotent because it can be seen that $UG^{(i+1)}$ acts trivially on $G/G^{(i)}$. Therefore UG is unipotent group of class at most $d-1$. The class is exactly $d-1$ because UG contains the group of inner automorphism which is isomorphic to $G/\operatorname{Cen}(G)$ and this is unipotent group of class exactly $d-1$.

The exact sequence splits because GL_n is reductive group and UG is an unipotent group, since $\mathbb{R}$ is a field. □

Similar result holds for the automorphism group of the discrete subgroup $\Gamma(n,d)$ of $G(n,d)$.

LEMMA 2.2.3. *The automorphism group* $\mathrm{Aut}\,\Gamma(n,d)$ *satisfies the following exact sequence:*

$$1 \to \mathrm{IAut}\,\Gamma(n,d) \to \mathrm{Aut}\,\Gamma(n,d) \to \mathrm{GL}_n(\mathbb{Z}) \to 1,$$

where the group $\mathrm{IAut}\,\Gamma(n,d)$ *consist of all automorphisms, which act trivially on $\Gamma/[\Gamma,\Gamma]$. This is a nilpotent group of nilpotency class $d-1$. Note that the exact sequence does not split into semidirect product unless $d=1$ and Γ is abelian.*

PROOF. The group $\Gamma(n,d)$ is a nilpotent group and a lattice in $G(n,d)$. Therefore, its automorphism group can be embedded into $\mathrm{Aut}\,G(n,d)$.

Let us define $\mathrm{IAut}\,\Gamma(n,d) = \mathrm{Aut}\,\Gamma \cap UG$. Since UG is a normal subgroup of $\mathrm{Aut}\,G$, $\mathrm{IAut}\,\Gamma$ is a normal subgroup of $\mathrm{Aut}\,\Gamma$. The group $\mathrm{IAut}\,\Gamma$ is a lattice in UG, since it

is isomorphic to $\mathrm{Hom}(\Gamma/[\Gamma,\Gamma],[\Gamma,\Gamma])$ as a set. The quotient $\operatorname{Aut}\Gamma/\operatorname{IAut}\Gamma$ acts on $\Gamma/[\Gamma,\Gamma]\simeq\mathbb{Z}^n$, therefore it is a subgroup of $\mathrm{GL}_n(\mathbb{Z})$. It is the whole group because $\mathrm{GL}_n(\mathbb{Z})$ is generated by the elementary and diagonal matrices, and they can be easily lifted to automorphisms of Γ.

Unlike in the case of $G(n,d)$ this exact sequence does not split in general. In fact it splits only when Γ is abelian and the group $\operatorname{IAut}\Gamma$ is trivial. □

Since we are interested in whether the group $\operatorname{Aut}\Gamma(n,d)$ has Kazhdan property T, and property T is invariant under passage to finite index subgroups, in this chapter we will work mainly with the subgroup of index 2 in $\operatorname{Aut}\Gamma$ — $\operatorname{Aut}_1\Gamma(n,d)=\pi^{-1}(\mathrm{SL}_n(\mathbb{Z}))$ consisting of all automorphisms which act on the abelinization as matrices with determinant 1.

LEMMA 2.2.4. *The Zariski closure* $\operatorname{Aut}_1 G(n,d)$ *of the group* $\operatorname{Aut}_1\Gamma(n,d)$ *in* $\operatorname{Aut} G$ *is*

$$\operatorname{Aut}_1 G(n,d)=\mathrm{SL}_n(\mathbb{R})\ltimes UG(n,d).$$

It is a connected Lie group and the group $\operatorname{Aut}_1\Gamma$ *is a lattice in* $Aut^1G(n,d)$.

PROOF. The group $\operatorname{Aut}_1\Gamma(n,d)$ is a lattice in $\operatorname{Aut}_1 G(n,d)$ because $\operatorname{IAut}\Gamma(n,d)$ is a lattice in $UG(n,d)$ and $\mathrm{SL}_n(\mathbb{Z})$ is a lattice in $\mathrm{SL}_n(\mathbb{R})$. □

DEFINITION 2.2.5. Let us denote by $G^1(n,d)$ the Zariski closure of $\operatorname{Aut}\Gamma$ in $\operatorname{Aut} G$. This is a Lie group which has two connected components and is isomorphic to

$$G^1(n,d)=\mathrm{GL}_n^{\pm 1}(\mathbb{R})\ltimes UG(n,d).$$

2.3. Kazhdan Property T

Kazhdan property T originated from the representation theory of the Lie groups (see [**14**]). For a group G, let G^* denote the unitary dual of the group G consisting of all irreducible unitary representations of G up to isomorphism. It is possible to put topology (called Felt topology) on the set G^*, saying that two representations $(\rho_0, \mathcal{H}_0)$ and $(\rho_1, \mathcal{H}_1)$ are close if it is possible to find unitary vectors $v_0 \in \mathcal{H}_0$ and $v_1 \in \mathcal{H}_1$ such that the matrix coefficients $\langle v_0, gv_0\rangle$ and $\langle v_0, gv_0\rangle$ are close (on the compact subsets of G) as functions of g. In this terminology it is said that G has Kazhdan property T, if and only the trivial representation is an isolated point in G^* with respect to this topology.

Here we will use an equivalent definition which is more appropriate for our goals.

Definition 2.3.1. A topological group G, generated by a compact set Q, is said to have Kazhdan property T if there exists a constant ϵ such that any unitary representation $(\rho, \mathcal{H})$ of the group G, which contains a unit vector v such that $||\rho(g)v - v|| \leq \epsilon$ for any $g \in Q$, contains a G invariant vector. The maximal ϵ with this property is called the Kazhdan constant of G with respect to Q and is denoted by $\mathcal{K}(G, Q)$. This is equivalent to following expression

$$\mathcal{K}(G, Q) = \inf_{(\rho,\mathcal{H})\in G^*} \inf_{\substack{v\in\mathcal{H} \\ ||v||=1}} \sup_{g\in Q} ||\rho(g)v - v||,$$

where the first infimum is over all unitary irreducible representations of the group G.

Remark 2.3.2. The above definition says when the pair (G, Q) has Kazhdan property T. It can be shown that having property T depends only on the group

G and its topology, but the value of the Kazhdan constant depends also on the generating set Q.

As an immediately corollary of the remark we have

COROLLARY 2.3.3. Compact (or finite) groups have property T, because for the generating set Q we can take G itself. In fact it can be shown that for any group G we have $\mathcal{K}(G, G) \geq \sqrt{2}$.

REMARK 2.3.4. Let G' be a finite index subgroup of a group G. Using induction and restriction of representations from G' to G and vice versa it is easy to see that the group G' has Kazhdan property T if and only if the group G has property T.

This result was generalized by Kazhdan in the case of Lie groups and their lattices.

THEOREM 2.3.5 (Kazhdan). *A lattice in a Lie group has property T if and only if the Lie group has property T.*

REMARK 2.3.6. The groups $\mathbb{Z}$ and $\mathrm{SL}_2(\mathbb{Z})$ do not have Kazhdan property T — the unitary dual of the group $\mathbb{Z}$ is isomorphic to the circle $\mathbb{R}/\mathbb{Z}$ and the Felt topology on it coincides with the usual topology on this set, therefore there are no isolated points in the unitary dual of $\mathbb{Z}$. The group $\mathrm{SL}_2(\mathbb{Z})$ does not have property T because it contains a free group as a finite index subgroup.

COROLLARY 2.3.7. If G has property T then any quotient also have T, in particular a group G can not have property T is it has infinite abelian quotient.

Proving that a group G has a property T directly using the above definition requires detailed knowledge of the unitary dual G^*. In [**14**] Kazhdan avoided this problem for simple Lie groups.

THEOREM 2.3.8 (Kazhdan). *Let G be simple Lie group over $\mathbb{R}$, then G has property T if the rank of G is at least 2.*

REMARK 2.3.9. The situation with simple Lie groups of rank 1 is more complicated - some groups for example $F_{4,-22}$ have property T and others like $\mathrm{SL}_2(\mathbb{R})$ do not.

An obvious obstruction for a Lie group G to have property T is to have $\mathbb{R}$ as a quotient (because the group $\mathbb{R}$ like the group $\mathbb{Z}$ does not have T). It is interesting that for Lie groups with big semi simple part this is the only obstruction. In order to show this we need to state the following result proved by Wang in [**22**] as a consequence of the so called Maunter Phenomena.

THEOREM 2.3.10 (Wang). *Let G be a Lie group over $\mathbb{R}$. Let S denotes the reductive part and N denotes the unipotent part of the group G. If the group N coincides with $[N, S]$ then there exist finitely many S orbits O_i, $i = 1, \dots, k$, in N, such that any element n in N can be written as $n = o_1 o_2 \dots o_k$, where $o_i \in O_i$ or $o_i = 1$.*

PROOF. We will just sketch the proof. For details we refer the reader to the original paper by Wang [**22**].

The proof of this theorem is by induction of the nilpotency class of N. The base case is when the unipotency class is 0 and the group N is trivial. In this case there is nothing to prove.

By induction every element in $N/\operatorname{Cen}(N)$ can be written as a product of finitely many S orbits in $N/\operatorname{Cen}(N)$. We can lift these orbit to orbits in N, which implies that every element in N is a product of representatives of these orbits up to an element in the center of N.

Therefore in order to prove that the same is true for N, we only need to show that $\operatorname{Cen}(N)$ can be written as a product of finitely many S orbits. The abelian group $\operatorname{Cen}(N)$ decomposes as a direct product of finite number of subgroups N_i, which are simple S modules. If the action of S on N_i is not trivial, then $N_i \setminus 0$ consists of a single S orbit. Clearly every element in the product of nontrivial N_i-es can be written as a product of elements in some finite number of S orbits.

The condition $N = [N, S]$ implies that $N_k \subset [N, N]$ for any k such that N_k is isomorphic to the trivial S module. Using that induction hypothesis we can express every element in $N/\operatorname{Cen}(N)$ as a product of finite number of S orbits and therefore the same is true for the commutator of any two elements in N. This shows that we can find S orbits O_j such that every element in $\operatorname{Cen}(N)$ can be written as a product of elements $o_j \in O_j$, which completes the proof of the induction step. $\square$

We can use this result to find a necessary and sufficient condition for a group with big nilpotent radical to have Kazhdan Property T.

THEOREM 2.3.11 (Wang). *Let G be a Lie group over G. Let S denotes the reductive part and N denotes the unipotent part of the group G. Then the following two conditions are equivalent*

(1) *G has property T;*
(2) *S has property T and $N/[S, N]$ is compact.*

PROOF. Clearly 1 implies 2, because S is a quotient of G and therefore has T, and the quotient $N/[S, N]$ can be embedded into $G/[G, G]$ therefore it is compact, because the only abelian groups which have property T are the compact ones.

We will only sketch why 2 implies 1. Let construct a compact generating set K of the group G consisting of a compact generating set of the reductive part S together with some compact subset of N which projects onto $N/[S,N]$. We need also to add to K some generating set of $[S,N]$, let $G' = S \ltimes [S,N]$ this group satisfies the conditions of Theorem 2.3.10. Therefore there exist a finite number of S orbits O_i such that every element in $[S,N]$ is a product of representatives of these orbits. Let us include in K one representative from each orbit. It is easy to see that the set K constructed in this way is a compact generating set of G.

Let us take a representation $(\pi,\mathcal{H})$ of the group G and suppose that v is an ϵ invariant unit vector in $\mathcal{H}$ for epsilon small enough, with respect to the generating set of K.

Using the fact that S has property T we can pass to a new vector w which is ϵ' invariant with respect to the set K but such that the group S acts trivially on w, where ϵ' depends only on ϵ and the Kazhdan constant of S.

The last condition implies that w is ϵ' invariant with respect to every element in the orbits O_k. By Theorem 2.3.10 every element in the group G can be expressed as a product of a fixed number of elements which almost preserve the vector w, therefore w is moved a distance less then 1 by any element in G if ϵ is sufficiently small. This shows that $\mathcal{H}$ contains invariant vectors, which proves that the group G has Kazhdan property T. □

2.4. Property T of $\operatorname{Aut}\Gamma(n,d)$

In [**15**] A. Lubotzky and I. Pak studied the product replacement algorithm on nilpotent groups. They proved that the working time of is algorithm is logarithmic using the fact that a special subgroup of the automorphism group $\operatorname{Aut}\Gamma(n,d)$ has

property T provided that $n \geq 3$. Since this subgroup is of infinite index they asked the question when the whole automorphism group has property T. Using the results from previous sections we can answer this question.

THEOREM 2.4.1. *The automorphism group* $\operatorname{Aut}\Gamma(n,d)$ *of the free nilpotent group on* n *generators and prepotency class* d *has Kazhdan property* T *if and only if* $n \geq 3$ *and* $d \leq h(n)$*, where* $h(n)$ *is the function defined in definition 1.5.1. In particular it has property* T *if* $d \leq n(n-1)$ *and* $n \geq 3$*. If conjecture 1.5.2 holds that it will have property* T *for any* d*, provided that* $n \geq 3$*.*

PROOF. The condition $n \geq 3$ is necessary because the group $\operatorname{Aut}\Gamma(2,d)$ has $\mathrm{GL}_2(\mathbb{Z})$ as a quotient and this group does not have property T, since it is virtually free.

The main step in the proof of the theorem is the following lemma

LEMMA 2.4.2. *The following statements are equivalent*

(1) $\operatorname{Aut}\Gamma(n,d)$ *has property* T*;*

(2) $\operatorname{Aut}_1\Gamma(n,d)$ *has property* T*;*

(3) $\operatorname{Aut}_1 G(n,d)$ *has property* T*;*

(4) $\operatorname{Aut}_1 G(n,d)$ *does not have* $\mathbb{R}$ *as a quotient and* $n \geq 3$*;*

(5) *the abelinization of the Lie algebra* $\mathfrak{sl}_n + UL(n,d)$ *is trivial and* $n \geq 3$*;*

(6) $n \geq 3$ *and* $(UL(n,d)/[UL(n,d),UL(n,d)])^{\mathfrak{sl}} = 0$*.*

PROOF. Many of the equivalences among the above statements are easy to prove. Later we will use that 1 is equivalent to 6 in order to prove Theorem 2.4.1.

1 is equivalent to 2, because $\operatorname{Aut}_1\Gamma(n,d)$ is a subgroup of index 2 in $\operatorname{Aut}\Gamma(n,d)$ and we can use remark 2.3.4;

2 is equivalent to 3, by Kazhdan Theorem 2.3.5 because $\operatorname{Aut}_1 \Gamma(n,d)$ is a lattice in $\operatorname{Aut}_1 G(n,d)$;

3 and 4 are equivalent by the Maunter phenomenon (Theorem 2.3.11), because the group $\operatorname{Aut}_1 G(n,d)$ is connected, simply connected and can not have any non trivial compact quotients;

4 and 5 are the same by Lie algebra arguments;

5 and 6 because the abelinization of the Lie algebra $\mathfrak{sl}_n + UL(n,d)$ is equal to $UL(n,d)/[UL(n,d),UL(n,d)]^{\mathfrak{sl}}$. □

By definition $h(n)$ is the smallest degree of homogeneous element in the factor space $UL[V]/[UL[V],UL[V]]^{\mathfrak{sl}}$. By construction the algebra $UL(n,d)$ is a quotient of $UL[V]$ containing all homogenous components of degree less then d. Therefore the space $UL(n,d)/[UL(n,d),UL(n,d)]^{\mathfrak{sl}}$ is trivial if and only if $d \leq h(n)$, which completes the proof of the theorem.

□

2.5. The center of $\operatorname{Aut}\Gamma(n,d)$

In this section we give a proof of a result by Formanek[**6**], which describes the center of the automorphism group of $\Gamma(n,d)$. We will use this result latter in Chapter 4, where we will describe the automorphism tower of the group $\Gamma(n,d)$.

Theorem 2.5.1 (Formanek). *The center of the group* $\operatorname{Aut}\Gamma(n,d)$ *is not trivial if and only if* $2n|d-1$.

Proof. Every element in the center of the group $\operatorname{Aut}\Gamma$ should lie in the center of the group $\operatorname{IAut}\Gamma$. We have seen that

$$\operatorname{Cen}(\operatorname{IAut}\Gamma) = \operatorname{Hom}(\Gamma/[\Gamma,\Gamma], \operatorname{Cen}\Gamma).$$

An element $\phi \in \operatorname{Cen}(\operatorname{IAut}\Gamma)$ lie in the center of the group $\operatorname{Aut}\Gamma$ iff it is preserved by the action of $\mathrm{GL}_n(\mathbb{Z})$ on this space. Since the action of $\mathrm{GL}_n(\mathbb{Z})$ in these spaces is polynomial of degree 1 and d respectively, a necessary condition for existence of invariant elements is that $2n|d-1$.

Therefore we only need to show that if $2n|d-1$ there exists $\mathrm{GL}_n(\mathbb{Z})$ invariant elements in the $\operatorname{Hom}(\Gamma/[\Gamma,\Gamma], \operatorname{Cen}\Gamma)$. If $d > 4n$ or $n \geq 3$ this is true because that are $\mathrm{GL}_n(\mathbb{Z})$ invariant elements in the homogeneous component of the free Lie algebra of degree $2kn$ for $k \geq 2$ or $n > 3$, and the homogeneous component of the free Lie algebra can be embedded in the above space of homomorphisms. In the cases $n = 2$, $d = 5$ and $n = 3$, $d = 7$ we need to verify that in the homogeneous component of the free Lie algebra of degree d there is a $\mathfrak{sl}$ submodule isomorphic to the simple module corresponding to the partition $[2k+1, 2k^{n-1}]$. $\square$

CHAPTER 3

Derivation Tower of Free Nilpotent Lie algebras

In this chapter we study the derivation tower of free nilpotent Lie algebras. It is known that in the case of Lie groups there is a close connection between the automorphism group of the Lie group and the derivation algebra of its Lie algebra. Using this connection is natural first one to study the derivation tower of the Lie algebra of a group and after that to try to lift the results to the automorphism tower of the Lie group.

It turns out (see Theorem 3.4.1) that the derivation tower of free nilpotent Lie algebras is very short — it has height at most 3 and similar result holds for the automorphism tower of the free unipotent group.

Unfortunately the situation with the free nilpotent group is different. One explanation for this difference is that in the ring $\mathbb{Z}$ there are just 2 invertible elements and therefore the automorphism group of a lattice in a Lie group is much smaller then the automorphism group of the Lie group. A very illustrative example is $\mathbb{Z}^n \subset \mathbb{R}^n$. Their automorphism groups are $\mathrm{GL}(\mathbb{Z})$ and $\mathrm{GL}(\mathbb{R})$ respectively. If one wants to use some Lie group to study the group $\mathrm{GL}(\mathbb{Z})$ it is better to use $\mathrm{SL}^{\pm 1}(\mathbb{R})$ not the whole group $\mathrm{GL}(\mathbb{R})$.

Having in mind this discrepancy, we define the algebra of restricted derivations (see definition 3.5.1) and construct the tower of restricted derivations. Studying this tower, we found that the behavior of the tower is different depending whether the

first algebra in this tower $\operatorname{Der}_0 L[V,d]$ has trivial center or not. The main results in this chapter are Theorems 3.9.1 and 3.10.1, which state that the tower of restricted derivations of the free nilpotent Lie algebra stabilizes after finitely many steps.

In the next chapter 4 we will lift these result to the automorphism tower of the free nilpotent group.

3.1. Derivation tower of Lie algebras

If we have a Lie algebra L we can construct its derivation algebra $\operatorname{Der} L$, which comes with a natural map $\operatorname{ad} : L \to \operatorname{Der} L$.

Definition 3.1.1. A Lie algebra L is called complete if it has trivial center and no outer derivations. This is equivalent to the map ad being an isomorphism.

Iterating the procedure of constructing the derivation algebra, we can define the derivation tower of the Lie algebra L.

Definition 3.1.2. Let L be a (finite dimensional) Lie algebra, by a derivation tower of the algebra L we mean, the sequence

$$\operatorname{Der}^0 L \to \operatorname{Der}^1 L \to D^2 L \to \cdots \to \operatorname{Der}^n L \to \dots,$$

where $\operatorname{Der}^0 L = L$ and for any i we have $\operatorname{Der}^{i+1} L = \operatorname{Der}(\operatorname{Der}^i L)$. The maps ad_i : $\operatorname{Der}^i L \to \operatorname{Der}^{i+1} L$ comes form the inner derivations of the algebra $\operatorname{Der}^i L$. In general this maps are neither injective nor surjective.

Remark 3.1.3. If the algebra $\operatorname{Der}^i L$, for some i, has a trivial center then the map ad_i is an injection. By Remark 3.2.2) we have that the algebra $\operatorname{Der}^{i+1} L$ also has a trivial center and therefore by induction all maps ad_k are injections for every $k \geq i$.

Definition 3.1.4. We say that the derivation tower of the algebra L stabilizes at level k (or after k steps) if the map ad_k is an isomorphism. We also say that the tower weakly stabilizes at level k, iff $\operatorname{Der}^k L \simeq \operatorname{Der}^{k+1} L$.

Example 3.1.5. Derivation tower of the algebra $\mathfrak{sl}_n$. The algebra $\mathfrak{sl}$ has trivial center and has no outer derivations. Therefore the derivation tower of $\mathfrak{sl}_n$ stabilizes at the level 0 and the tower is

$$\mathfrak{sl}_n \to \mathfrak{sl}_n \to \cdots \to \mathfrak{sl}_n \to \ldots.$$

Example 3.1.6. Derivation tower of $\mathbb{R}$. The derivation algebra of $\mathbb{R}$ is also $\mathbb{R}$, but the map ad is trivial because $\mathbb{R}$ is abelian Lie algebra. This shows that the derivation tower of $\mathbb{R}$ is

$$\mathbb{R} \to \mathbb{R} \to \cdots \to \mathbb{R} \to \ldots,$$

and all maps ad_k are trivial. Therefore the tower does not stabilize, but it stabilizes weakly at level 0.

3.2. Derivation Algebras

First let us prove several lemmas about derivation algebras, their centers and centralizers of certain ideals. We will use these lemmas in the next section to describe the derivation algebra of $\operatorname{Der} L[V, d]$.

Lemma 3.2.1. *Let L be a Lie algebra. Then the set of all inner derivations* $\operatorname{ad} L$, *is an ideal in the algebra* $\operatorname{Der} L$ *and its centralizer is isomorphic to*

$$\operatorname{Cen}_{\operatorname{Der} L}(\operatorname{ad}(L)) = \operatorname{Hom}(L/[L, L], \operatorname{Cen}(L)).$$

Proof. Let $d \in \operatorname{Der} L$ be a derivation of L, which commutes with all inner derivations. Then

$$0 = [d, \operatorname{ad} x](y) = d([x, y]) - [x, d(y)],$$

but we have that

$$d([x, y]) = [d(x), y] + [x, d(y)].$$

From these equalities it follows that $[d(x), y] = 0$, for all $x, y \in L$. Therefore $d(x) \in \operatorname{Cen}(L)$ for all $x \in L$. From the Leibnitz rule, it follows that $d([x, y]) = 0$, i.e., $d_{|[L,L]} = 0$. This shows that we can view d as a map from $L/[L, L]$ to $\operatorname{Cen}(L)$. This is an isomorphism, because every element in $\operatorname{Hom}(L/[L, L], \operatorname{Cen}(L))$ can be extended to a derivation of L, by letting it act trivially on the commutator subalgebra, and the direct computation verifies that this defines a derivation which commutes with all inner derivations. □

Remark 3.2.2. If the algebra L has a trivial center then the centralizer of the ideal of inner derivations is trivial, in particular we have that the algebra $\operatorname{Der} L$ also has a trivial center.

Lemma 3.2.3. *Let L be a Lie algebra. We can consider the inner derivations as a map* $\operatorname{ad} : L \to \operatorname{Der} L$. *Let D be a derivation of* $\operatorname{Der} L$ *(i.e. an element in* $\operatorname{Der}(\operatorname{Der} L)$*), which acts trivially on* $\operatorname{ad}(x)$ *for all $x \in L$. Then $(D(d))(x) \in \operatorname{Cen}(L)$ for all $x \in g$ and $d \in \operatorname{Der} L$, i.e.,* $\operatorname{Im} D \subset \operatorname{Cen}_{\operatorname{Der} L}(\operatorname{ad}(L))$.

Proof. Let us note that $[d, \operatorname{ad}(x)] = \operatorname{ad}(d(x))$ for all $d \in \operatorname{Der} L$. Applying the derivation D to both sides yields

$$0 = D(\operatorname{ad}(d(x))) = D[d, \operatorname{ad}(x)] = [D(d), \operatorname{ad}(x)] + [d, D(\operatorname{ad}(x))] = \operatorname{ad}(D(d)(x)),$$

therefore $D(d)(x) \in \operatorname{Cen} g$ for all x. From lemma 3.2.1 it follows that $\operatorname{Im} D \subset \operatorname{Cen}_{\operatorname{Der} L}(\operatorname{ad}(L))$. $\square$

COROLLARY 3.2.4. Let L be a Lie algebra with trivial center. The adjoint action gives a map $\operatorname{ad} : L \to \operatorname{Der} L$. Let D be a derivation of $\operatorname{Der} L$ (i.e. an element in $\operatorname{Der}(\operatorname{Der} L)$), which acts trivially on $\operatorname{ad}(x)$ for all $x \in L$. Then $D = 0$. This shows that any derivation of $\operatorname{Der} L$ is determined by its restriction to $\operatorname{ad}(L)$.

Lemma 3.2.3 can be generalized to derivation acting trivially on ideals in Lie algebras.

LEMMA 3.2.5. *Let $D \in \operatorname{Der} L$ be a derivation which acts trivially on the ideal I, i.e., $D(i) = 0$ for all $i \in I$. Then $\operatorname{Im} D \in \operatorname{Cen}_L(I)$.*

PROOF. Let $g \in L$ and $i \in I$. Since I is an ideal we have $[g, i] \in I$, if we apply the derivation D to both sides we get $[D(g), i] + [g, D(i)] = 0$, which gives that $[D(g), i] = 0$, i.e., $D(g) \in \operatorname{Cen}_L(I)$ for any $g \in L$. $\square$

3.3. Chevalley Theorem

In [**3**] Chevalley stated that if $\mathfrak{g}$ is a finite dimensional Lie algebra, over a field of characteristic 0, with trivial center, then its derivation tower stabilizes after finitely many steps. This result was generalized by Schenkman to the case of positive characteristic see [**19**]. However, this result can not be generalized to arbitrary finite dimensional Lie algebra, because there are examples (like Example 3.1.6) of Lie algebras with non stabilizing derivation towers.

Chevalley's note contains only a sketch of the proof, in this section we give a detailed proof of this result, because in the next sections we will make several constructions similar to the ones in this proof.

The following technical lemma is a key point in the proof.

LEMMA 3.3.1. *Let $\mathfrak{g}$ be a Lie algebra with trivial center, which has the following decomposition*

$$\mathfrak{g} = \mathfrak{s} + \mathfrak{a} + \mathfrak{n},$$

where $\mathfrak{s}$ is a semi simple Lie algebra, $\mathfrak{a}$ is abelian and $\mathfrak{n}$ is nilpotent, such that the following properties are satisfied:

- $\mathfrak{a}$ *commutes with* $\mathfrak{s}$*;*
- $\mathfrak{a}$ *acts diagonally on* $\mathfrak{n}$*;*
- $\mathfrak{n}$ *is an ideal in* $\mathfrak{g}$*;*
- $\mathfrak{n}$ *is generated by* $[\mathfrak{s}, \mathfrak{n}]$ *and* $[\mathfrak{a}, \mathfrak{n}]$ *as a Lie subalgebra.*

Let $\operatorname{Der} \mathfrak{g}$ *denotes the derivation algebra of* $\mathfrak{g}$*. Since* $\mathfrak{g}$ *has no center we can consider* $\mathfrak{g}$ *as an ideal of* $\operatorname{Der} \mathfrak{g}$*. Let d be a derivation from* $\mathfrak{g}$ *to* $\operatorname{Der} \mathfrak{g}$*, i.e. a linear map which satisfies the equality*

$$d[x, y] = [d(x), y] + [x, d(y)]$$

for any $x, y \in \mathfrak{g}$. Then $\operatorname{Im} d \subset \mathfrak{g}$, i.e., $d \in \operatorname{Der} \mathfrak{g}$.

PROOF. We have that $d[x, y] \in \mathfrak{g}$ for any $x, y \in \mathfrak{g}$, because $\mathfrak{g}$ is an ideal in $\operatorname{Der} \mathfrak{g}$. Therefore, $d[\mathfrak{g}, \mathfrak{g}] \in \mathfrak{g}$. From the properties of the decomposition of $\mathfrak{g}$ it follows that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{s} + \mathfrak{n}$. Therefore we only need to show that $d(\mathfrak{a}) \subset \mathfrak{g}$.

The Lie algebra $\mathfrak{a}$ acts diagonally on $\mathfrak{g}$ and Der $\mathfrak{g}$. This action defines a $\mathfrak{a}^*$ gradation on both $\mathfrak{g}$ and Der $\mathfrak{g}$. Without loss of generality, we may assume that the derivation d is homogeneous with respect to the gradation of Der $\mathfrak{g}$.

Suppose that the degree $\lambda \in \mathfrak{a}^*$ of d is different from zero. Then $d(\mathfrak{a}) \subset V$ for some nontrivial $\mathfrak{a}$ module V, such that $[a, v] = \lambda(a)v$ for any $a \in \mathfrak{a}$ and $v \in V$. But if $\lambda \neq 0$ then $[\mathfrak{a}, V] = V$, which implies that $V \subset \mathfrak{g}$, since $\mathfrak{a} \subset \mathfrak{g}$ and $\mathfrak{g}$ is an ideal in Der $\mathfrak{g}$. Thus we have shown that $d(\mathfrak{a}) \subset \mathfrak{g}$, which implies that $d(\mathfrak{g}) \subset \mathfrak{g}$.

Now suppose that the degree of d is zero. Let v be an eigenvector of $\mathfrak{a}$ in Der $\mathfrak{g}$ corresponding to the eigenvalue $\lambda \in \mathfrak{a}^*$. By the homogeneity of d it follows that $d(v)$ is also an eigenvector for $\mathfrak{a}$, corresponding to the same eigenvalue. Thus,

$$d[a, v] = d(\lambda(a)v) = \lambda(a)d(v)$$

and

$$d[a, v] = [d(a), v] + [a, d(v)] = [d(a), v] + \lambda(a)d(v).$$

Therefore, $[d(a), v] = 0$, for any $a \in \mathfrak{a}$ and any eigenvector v. This implies that $[d(a), x] = 0$ for any $x \in$ Der $\mathfrak{g}$ because $\mathfrak{a}$ acts diagonally on Der $\mathfrak{g}$. From Remark 3.2.2 it follows that the center of the algebra Der $\mathfrak{g}$ is trivial, which implies that $d(a) = 0$ for all $a \in \mathfrak{a}$. Thus we have $d(\mathfrak{a}) = 0 \subset \mathfrak{g}$, i.e. $d(\mathfrak{g}) \subset \mathfrak{g}$, which completes the proof of the lemma. □

Corollary 3.3.2. Let $\mathfrak{g}$ be a Lie algebra which satisfies the conditions of the lemma, then its derivation algebra is complete, i.e., $\mathrm{Der}(\mathrm{Der}\,\mathfrak{g}) = \mathrm{Der}\,\mathfrak{g}$.

Theorem 3.3.3. *Let $\mathfrak{g}_\circ$ be a finite dimensional Lie algebra over a field of characteristic* 0*, with a trivial center. Then the derivation tower* $\mathrm{Der}^n(\mathfrak{g}_\circ)$ *of* $\mathfrak{g}_\circ$ *stabilizes after finitely many steps.*

PROOF. The algebra $\mathfrak{g}_o$ has the following decomposition

$$\mathfrak{g}_o = \mathfrak{s} + \mathfrak{a} + \mathfrak{n}_o,$$

where $\mathfrak{s}$ is a semi simple Lie algebra, $\mathfrak{a}$ is abelian and $\mathfrak{n}_o$ is nilpotent, such that the following properties are satisfied:

$\mathfrak{a}$ commutes with $\mathfrak{s}$;

$\mathfrak{a}$ acts diagonally on $\mathfrak{n}_o$;

$\mathfrak{n}_o$ is the maximal nilpotent ideal of $\mathfrak{g}_o$.

Let $\mathfrak{n}$ be the subalgebra of $\mathfrak{n}_o$ generated by the elements $[\mathfrak{s}, \mathfrak{n}_o]$ and $[\mathfrak{a}, \mathfrak{n}_o]$. Let $\mathfrak{g}$ denote the subalgebra $\mathfrak{g} = \mathfrak{s} + \mathfrak{a} + \mathfrak{n}$.

LEMMA 3.3.4. *The subalgebra $\mathfrak{g}$ is an ideal in $\mathfrak{g}_o$.*

PROOF. It suffices to show that $[[a, n_1], n_2] \in \mathfrak{n}$ for any $a \in \mathfrak{a}$ and $n_1, n_2 \in \mathfrak{n}_o$, and that $[[s, n_1], n_2] \in \mathfrak{n}$ for any $s \in \mathfrak{s}$ and $n_1, n_2 \in \mathfrak{n}_o$. Let us prove the first one. Without loss of generality, we may assume that n_2 is an eigenvector for $\mathfrak{a}$. If the corresponding eigenvalue is zero, then $[\mathfrak{a}, n_2] = 0$, therefore $[[a, n_1], n_2] = [a, [n_1, n_2]] \in [\mathfrak{a}, \mathfrak{n}_o] \subset \mathfrak{n}$. If the corresponding eigenvalue is not zero then $n_2 = [a_1, n_2]$ for some $a_2 \in \mathfrak{a}$, which gives that $[[a, n_1], n_2] = [[a, n_1], [a_1, n_2]] \in \mathfrak{n}$. The proof that $[[s, n_1], n_2] \in \mathfrak{n}$ for any $s \in \mathfrak{s}$ and $n_1, n_2 \in \mathfrak{n}_o$ is similar.

$\square$

LEMMA 3.3.5. *The centralizer of $\mathfrak{g}$ in $\mathfrak{g}_o$ is trivial.*

PROOF. Let $J = \{x \in \mathfrak{g}_o \mid [x, \mathfrak{g}] = 0\}$ be the centralizer of $\mathfrak{g}$. It is easy to see that $J \subset \mathfrak{n}_o$ and J is an ideal in $\mathfrak{g}_o$. If J is non trivial then $J \cap \operatorname{Cen} \mathfrak{n}_o \neq \{0\}$, because in the nilpotent Lie algebra every nontrivial ideal has nontrivial intersection with the

center. It is easy to see that $J \cap \operatorname{Cen} \mathfrak{n}_\circ \subset \operatorname{Cen} \mathfrak{g}_\circ = \{0\}$, which is a contradiction. Therefore, $J = \{0\}$, which proves the lemma. $\square$

The above two lemmas allows us to identify the algebra $\mathfrak{g}_\circ$ with some sub algebra of Der $\mathfrak{g}$. Let $D \in \operatorname{Der}(\mathfrak{g}_\circ)$. Then the restriction of D to $\mathfrak{g}$, is a derivation from $\mathfrak{g}$ to $\mathfrak{g}_\circ \subset$ Der $\mathfrak{g}$. By Lemma 3.3.1 we have that $D_{|\mathfrak{g}} = D' \in$ Der $\mathfrak{g}$. We can consider the D' as a derivation from $\mathfrak{g}_\circ$ to Der $\mathfrak{g}$. By Lemma 3.2.1 we have that $D - D' = 0$ (in Der $\mathfrak{g}$), which allows us to identify $\operatorname{Der}(\mathfrak{g}_\circ)$ as a sub algebra of Der $\mathfrak{g}$. Now by simple induction we have

$$\mathfrak{g} \subset \mathfrak{g}_\circ \subset \operatorname{Der}^1(\mathfrak{g}_\circ) \subset \cdots \subset \operatorname{Der}^n(\mathfrak{g}_\circ) \subset \cdots \subset \operatorname{Der} \mathfrak{g}.$$

Since the algebra Der $\mathfrak{g}$ is finite dimensional, the above sequence has to stabilize at some point. Therefore, the derivation tower of $\mathfrak{g}_\circ$ stabilizes. $\square$

REMARK 3.3.6. This proof also gives us a bound on the height of the derivation tower — the height is less than $\dim \operatorname{Der} \mathfrak{g} - \dim \mathfrak{g}_\circ$.

3.4. Derivation tower of free nilpotent Lie algebras

Let us describe the derivation tower of the free nilpotent Lie algebra $L[V, d]$. We can not use directly the Chevalley's Theorem because the algebra $L[V, d]$ has nontrivial center. However we can apply Chevalley's result to its derivation algebra which does not have center. In chapter 1, we showed that its derivation algebra has a decomposition

$$\operatorname{Der} L[V, d] = \mathfrak{gl}_n + UL[V, d] = \mathfrak{sl}_n + \mathbb{R}.1 + UL[V, d]$$

This decomposition is of the same form as the one needed for Lemma 3.3.1. Using the fact that 1 acts as a degree derivation on UL, which is positively graded, we have

$[1, UL] = UL$. Therefore the conditions in the technical lemma 3.3.1 are satisfied, which gives us that the derivation algebra of $\operatorname{Der} L[V,d]$ is complete. This shows that the derivation tower of the algebra $L[V,d]$ stabilizes at most at the third level.

Let us describe the derivation algebra of $\operatorname{Der} L[V,d]$ in detail. Now let us assume that $d \geq 2$. Without loss of generality we may assume that any outer derivation d acts trivially on 1, $\mathfrak{sl}_n$ and $\operatorname{ad}(V)$ (see next section). Therefore d preserves the grading and the $\mathfrak{sl}_n$ module structure on the UL. Using the fact that d acts trivially on the space $\operatorname{ad}(V)$, it follows that d kills every element of the form $\operatorname{ad}(f)$ for some $f \in L[V,d]$. The last condition implies that $\operatorname{Im} d \subset \operatorname{Cen} UL$. This argument gives that if $d \geq 3$ there is an isomorphism between the space of outer derivations of the algebra $\operatorname{Der} L[V,d]$ and the space

$$\operatorname{Hom}_{\mathfrak{sl}}\left(U_{\mathrm{gen}} \cap UL^{(d-1)}, UL^{(d-1)}\right).$$

In particular the set of outer derivations is not trivial because $U_{\mathrm{gen}} \cap UL^{(d-1)}$ is not empty (it contains at least the $\mathfrak{sl}$ module generated by $\mathrm{AD}_{x_1^n}$).

In the case $d = 2$ situation is a bit more complicated because $\operatorname{ad}(V) \subset U_{\mathrm{gen}} \cap UL^{(d-1)}$. If $n \geq 3$ then $U_{\mathrm{gen}} \cap UL^{(1)} = UL^{(1)}$ is a sum of two simple $\mathfrak{sl}$ modules, therefore the space of outer derivations is one dimensional. In the case $n = 2$ we have that $UL^{(1)} = \operatorname{ad}(V)$, therefore there are no outer derivations.

Putting these arguments together we can describe the derivation tower of the algebra $L[V,d]$.

THEOREM 3.4.1. *a) If the nilpotency class d is at least 3 or the number of generators is at least 3 then the derivation tower of the free nilpotent Lie algebra $L[V,d]$*

is

$$L[V,d] \to \mathfrak{gl} + UL[V,d] \to \mathfrak{gl} + UL[V,d] + \mathrm{Hom}_{\mathfrak{sl}}\left(U^+_{\mathrm{gen}} \cap UL^{(d-1)}, UL^{(d-1)}\right),$$

and all other terms are equal to the $\mathrm{Der}^2 L[V,d]$*, i.e., the tower stabilizes at the second level.*

b) If the nilpotency class is $d = 2$ *and* $n = 2$ *then the tower stabilizes at the first level.*

3.5. Algebra of restricted derivations

Our goal is to use the description of the derivation tower of the algebra $L[V,d]$ to obtain information about the automorphism tower of the free nilpotent group. In chapter 2 we saw the ‘Lie algebra’ corresponding to the free nilpotent group is $L[V,d]$ and the ‘Lie algebra’ corresponding to its automorphism group is

$$\mathfrak{sl}_n + UL[V,d],$$

which is not the whole derivation algebra of $L[V,d]$, but a subalgebra of codimension 1. One reason for that is the fact the in the ring $\mathbb{Z}$ there are just two invertible elements, therefore the action of any automorphism on any characteristic quotient (as the abelinization of $\Gamma(n,d)$) should have ‘determinant’ plus or minus 1. This implies that ‘derivations’ which corresponds to such automorphism should has trace zero.

This argument suggest the instead of taking the whole derivation algebra we should take the subalgebra of all ‘traceless’ derivations.

DEFINITION 3.5.1. Let L be a Lie algebra and let $\operatorname{Der}_0 L$ denote the subalgebra of $\operatorname{Der} L$ consisting of 'totally traceless' derivations, i.e,

$$\operatorname{Der}_0 L = \{D \in \operatorname{Der} L \mid \operatorname{tr} D_I = 0 \text{ and } \operatorname{tr} D_{L/I} = 0 \text{ for any characteristic ideal } I \in L\}.$$

We will call $\operatorname{Der}_0 L$ the algebra of restricted derivations.

REMARK 3.5.2. Using the fact the commutator of any two linear maps has zero trace we can see that $[\operatorname{Der} L, \operatorname{Der} L] \subset \operatorname{Der}_0 L$. This implies that $\operatorname{Der}_0 L$ is a subalgebra $\operatorname{Der} L$. We will call this subalgebra the algebra of restricted derivations of L.

REMARK 3.5.3. In general we do not have that $\operatorname{ad}(\mathfrak{g}) \subset \operatorname{Der}_0 \mathfrak{g}$. For example take $\mathfrak{g}$ to be the two dimensional Lie algebra with basis a, b and commutator relation $[a, b] = b$. In this case we have that $\operatorname{tr} \operatorname{ad}(a) = 1$, therefore $\operatorname{ad}(a) \notin \operatorname{Der}_0 \mathfrak{g}$.

In general any Lie algebra $\mathfrak{g}$ can be written uniquely as

$$\mathfrak{g} = \mathfrak{s} + \mathfrak{a} + \mathfrak{n},$$

where $\mathfrak{s}$ is semi simple, $\mathfrak{a}$ is abelian and the action of $\operatorname{ad}(\mathfrak{a})$ on $\mathfrak{g}$ is diagonalizable, and $\mathfrak{n}$ is nilpotent. We have that $\operatorname{ad}(\mathfrak{g}) \subset \operatorname{Der}_0 \mathfrak{g}$, if and only if $\mathfrak{a}$ is trivial or $\mathfrak{a}$ acts trivially on $\mathfrak{n}$.

However, if there exists a Lie group G over $\mathbb{R}$ and Zariski dense lattice Γ in G such that the Lie algebra of G is $\mathfrak{g}$ then the above condition is satisfied and we have $\operatorname{ad}(\mathfrak{g}) \subset \operatorname{Der}_0 \mathfrak{g}$.

The next lemma allows us to define the tower of restricted derivations

LEMMA 3.5.4. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, then*

$$\operatorname{ad}(\operatorname{Der}_0 \mathfrak{g}) \subset \operatorname{Der}_0(\operatorname{Der}_0 \mathfrak{g}).$$

PROOF. Let us decompose $\operatorname{Der}\mathfrak{g}$ as

$$\operatorname{Der}\mathfrak{g} = \mathfrak{s} + \mathfrak{a} + \mathfrak{n},$$

and consider its action on $\mathfrak{g}$. The action of $\mathfrak{a}$ is diagonalizable and defines an $\mathfrak{a}^*$ grading on $\mathfrak{g}$, which decomposes into homogeneous components as $\mathfrak{g} = \sum V^\mu$.

Let $\mathfrak{n}_\circ$ be the subalgebra of $\mathfrak{n}$ generated by $[\mathfrak{n}, \mathfrak{a}]$. The direct computation verifies that $\mathfrak{n}_\circ$ is an ideal in $\operatorname{Der}\mathfrak{g}$ and its homogeneous component of degree 0 acts nilpotently on any V^μ.

Therefore the space $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{n}_\circ(\mathfrak{g})$, has nonzero homogeneous components $\tilde{V}^\mu$ (under the action of $\mathfrak{a}$) for any μ such that V^μ is not zero. This allows us to construct an ideal I in $\mathfrak{g}$ which is invariant under all derivation such that $\operatorname{tr}_{\mathfrak{g}/I} \operatorname{ad}(a) \neq 0$ for any $a \in \mathfrak{a}$, which acts nontrivially on $\mathfrak{g}$. Therefore $\mathfrak{a} \not\subset D_0\mathfrak{g}$ and we have that

$$\operatorname{Der}_0 \mathfrak{g} = \mathfrak{s} + \mathfrak{n},$$

and by the previous remark we have that $\operatorname{ad}(D_0\mathfrak{g}) \subset \operatorname{Der}_0(D_0\mathfrak{g})$, which finishes the proof. $\square$

DEFINITION 3.5.5. Restricted derivation tower. Let L be an Lie algebra such that $\operatorname{ad}(L) \subset \operatorname{Der}_0 L$. Then we can define the restricted derivation tower of L as follows

$$L = \operatorname{Der}_0^0 L \to \operatorname{Der}_0^1 L \to \operatorname{Der}_0^2 L \to \cdots \to \operatorname{Der}_0^n L \to \dots$$

where $\operatorname{Der}_0^{n+1} L = \operatorname{Der}_0(\operatorname{Der}_0^n L)$ and the maps come from the inner derivations. We say the tower stabilizes at the level k if ad_k is an isomorphism from $\operatorname{Der}_0^k L \to D_0^{k+1} L$.

REMARK 3.5.6. Let $\mathfrak{g}$ be an algebra with trivial center. Then $\mathfrak{g} \subset \operatorname{Der}_0 \mathfrak{g}$ if $\mathfrak{a} = 0$ or if $\mathfrak{a}$ acts trivially on $\mathfrak{n}$ (here we use the decomposition from Theorem 3.3.3). In this case it is easy to check that the proof about the stabilization of the derivation tower carries on to the tower of restricted derivations without any modifications.

EXAMPLE 3.5.7. Restricted derivation tower of $\mathbb{R}^n$. The derivation algebra of $\mathbb{R}^n$ is $\mathfrak{gl}_n$, which can be written as $\mathfrak{sl}_n + \mathbb{R}.\,\mathrm{id}$. The subalgebra of restricted derivations consist only of $\mathfrak{sl}_n$, because id acts on $\mathbb{R}^n$ as an operator with trave $n \neq 0$. This gives that $\mathrm{Der}_0\,\mathbb{R}^n = \mathfrak{sl}_n$. All other algebras in the tower of restricted derivations of $\mathbb{R}^n$ are equal to $\mathfrak{sl}_n$, because the algebra $\mathfrak{sl}_n$ is complete and coincide with its derived algebra. Therefore the tower is

$$\mathbb{R}^n \to \mathfrak{sl}_n \to \mathfrak{sl}_n \to \cdots \to \mathfrak{sl}_n \to \cdots$$

and stabilizes at the first level.

Finally let us define the operator Hom^0 which is similar of the Der^0

DEFINITION 3.5.8. Let V, W and U are linear spaces such that we have a projection $\pi : W \to U$ and an inclusion $i : U \to V$. Then there is a natural projection form $\mathrm{Hom}(V, W)$ to $\mathrm{End}(U)$. Let us define the space $\mathrm{Hom}^0(V, W)$, consisting of all elements in $\mathrm{Hom}(V, W)$ whose projection in $\mathrm{End}(U)$ has zero trace for all invariant U-es which are naturally a subspaces of V and factor factor spaces of W.

3.6. Distinguished ideals in UL

Before describing the tower of restricted derivations of the algebra $L[V, d]$ we need to define several invariant ideals in the algebras $UL[V]$ and $UL[V, d]$. The Lie algebra $\mathfrak{sl}$ is simple and it acts naturally on UL, therefore we can split UL as a direct sum

$$UL[V] = UL^{[0]}[V] + \sum_{\lambda \neq [0]} UL^{\lambda}[V],$$

where the sum is over all nontrivial partitions containing less than n parts. Here, $UL^{[0]}$ is the maximal trivial $\mathfrak{sl}$ submodule of UL and UL^{λ} is the maximal submodule, which can be written as a sum of simple $\mathfrak{sl}$ submodules corresponding to the partition λ.

Let us denote $UL^+ = \sum UL^\lambda$, this is the maximal submodule, which can be written as a sum of nontrivial $\mathfrak{sl}$ submodules.

DEFINITION 3.6.1. Let us define the following submodules of $UL[V]$:

UL_∞ as the ideal in UL generated by $[UL^+, UL^+]$;

$UL^+_{(0)} = UL^+$ and $UL^+_{(k+1)} = [UL^+_{(k)}, UL^{[0]}]$;

$UL^\lambda_{(0)} = UL^\lambda$ and $UL^\lambda_{(k+1)} = [UL^\lambda_{(k)}, UL^{[0]}]$;

$UL_k = UL_\infty + UL^+_{(k)}$;

$UL^\lambda_k = UL_{k+1} + UL^\lambda_{(k)}$.

The submodules UL^λ are preserved under all automorphisms of the algebra $\mathfrak{sl} + UL$. Therefore all the ideals defined above are also invariant.

LEMMA 3.6.2. *The submodules UL_k and UL^λ_k form a descending sequence of characteristic ideals in the algebra* $\mathrm{Der}_0\, L[V]$*, i.e.,*

$$UL_0 \supset UL_1 \supset \cdots \supset UL_k \supset \cdots \supset UL_\infty$$

$$UL^\lambda_0 \supset UL^\lambda_1 \supset \cdots \supset UL^\lambda_k \supset \cdots \supset UL^\lambda_\infty.$$

PROOF. UL_k is an ideal in UL because, we have the inclusions $[UL_{(k)}, UL^+] \subset [UL^+, UL^+]$ and $[UL_{(k)}, UL^{[0]}] \subset UL_{(k+1)} \subset UL_{(k)} \subset UL_k$. These inclusions prove that for every k, UL_k is an ideal in UL. The proof that UL^λ_k are ideals is similar. $\square$

DEFINITION 3.6.3. Let us denote by $U_{\mathrm{gen},k} = UL_k/UL_{k+1}$ and $U^\lambda_{\mathrm{gen},k} = UL^\lambda_k/UL_{k+1}$. We use the subscript 'gen' because a set S generates the algebra UL if and only if its image in $U_{\mathrm{gen},0}$ generates that space. We also want to define

$$U^{[0]}_{\mathrm{gen}} = UL^{[0]}/\big([UL, UL] \cap UL^{[0]}\big).$$

These modules will play an important role in the description of the derivation tower of $L[V,d]$. Also it is convenient to take the image of $\mathrm{ad}(V)$ away from $U_{\mathrm{gen},0}$ and write $U_{\mathrm{gen},0} = \mathrm{ad}(V) + U^{+}_{\mathrm{gen},0}$.

Similarly, we can define $UL_k[V,d]$, $UL^{\lambda}_k[V,d]$ and so on, as the submodules (ideals) of $UL[V,d]$ constructed form the algebra $UL[V,d]$ in the same way as the other modules were constructed from the algebra UL. The submodules $UL_k[V,d]$ and $UL^{\lambda}_k[V,d]$ again form a descending sequence of characteristic ideals in the algebra $\mathrm{Der}_0\, L[V,d]$.

Theorem 3.6.4. *The sequences $UL_k[V,d]$ and $UL^{\lambda}_k[V,d]$ of characteristic ideals in $UL[V,d]$ stabilize after at most $1 + d/h(n)$ steps. In particular if Conjecture 1.5.2 is true (or $d \leq h(n)$), they stabilize at the second term, i.e., $UL_1[V,d] = UL_2[V,d]$.*

Proof. In the factor space $UL^{[0]}/\big([UL,UL] \cap UL^{[0]}\big)$ there are no homogeneous components of degree less then $h(n)$. By induction, it follows that in

$$\big(UL^{+}_{(k)} + [UL^{+}, UL^{+}]\big)/\big(UL^{+}_{k+1} + [UL^{+}, UL^{+}]\big)$$

there are no homogeneous component of degree less then $kh(n)$. Therefore if $k > d/h(n)$ then $UL^{+}_{(k)} \subset [UL^{+}, UL^{+}]$, which implies that $UL_k = UL_{k+1} = \cdots = UL_{\infty}$. $\square$

Remark 3.6.5. Theorem 1.4.11 gives that if $d \leq n(n-1)$ then $U_{\mathrm{gen},0}$ is generated by the images of the elements δ_k for $k = 1, \ldots, d-1$ and δ_- as an $\mathfrak{sl}$ module. In that case we have $UL_1[V,d] = UL_{\infty}[V,d]$ and all other modules $U_{\mathrm{gen},k}[V,d]$ are trivial for $k \geq 1$. The Conjecture 1.5.2 implies that all modules $U_{\mathrm{gen},k}$ for $k \geq 1$ are trivial.

Definition 3.6.6. We need to define other modules, which will take part in the description of the tower of restricted derivations of $L[V, d]$. Let us denote by

$$CUL_k^\lambda[V, d] = UL_k^\lambda[V, d] \cap \mathrm{Cen}(UL[V, d]),$$

and

$$CU_{\mathrm{gen},k}^\lambda[V, d] = CUL_k^\lambda[V, d]/CUL_{k+1}^\lambda[V, d]$$

the intersections of the modules UL_k^λ and $U_{\mathrm{gen},k}^\lambda$ with the center of the algebra $UL[V, d]$. These modules are ideals in the algebras UL and $\mathfrak{sl}+UL$ because they lie in the center of UL.

Later when we construct the derivation tower of UL, we will see that the spaces $\mathrm{Hom}_{\mathfrak{sl}}(U_{\mathrm{gen},k}^\lambda, CUL_k^\lambda)$ play an important role. It is important to notice that there is a natural projection from that space to $\mathrm{End}_{\mathfrak{sl}}(CU_{\mathrm{gen},k}^\lambda)$, because $CU_{\mathrm{gen},k}^\lambda$ is naturally a subspace of the domain and factor space of the image.

Remark 3.6.7. We can also put an associative algebra structure on the spaces $\mathrm{Hom}_{\mathfrak{sl}}(U_{\mathrm{gen},k}^\lambda, CUL_k^\lambda)$, by defining the product of two maps fg to be their composition thought the space $CU_{\mathrm{gen},k}^\lambda$, i.e.,

$$(fg)(x) = f(i(\pi(g(x)))),$$

where π is the projection from CUL_k^λ to $CU_{\mathrm{gen},k}^\lambda$ and i is the inclusion of $CU_{\mathrm{gen},k}^\lambda$ into $U_{\mathrm{gen},k}^\lambda$. Notice that the associative algebra $\mathrm{Hom}_{\mathfrak{sl}}(U_{\mathrm{gen},k}^\lambda, CUL_k^\lambda)$ does not have unit unless $U_{\mathrm{gen},k}^\lambda = CUL_k^\lambda$.

3.7. Normalizer of the ideal of inner derivations

Let us introduce a notation for the normalizer of the ideal of inner derivations in $UL[V, d]$. We will show that it coincides with the ideal of inner derivations if $n \geq 3$.

We will need this ideal in the description of the second derivation algebra of $L[V,d]$ (see section 3.8).

DEFINITION 3.7.1. Let $\widetilde{UL}[V,d]$ be the nilpotent part of the algebra of outer derivations of $L[V,d]$, i.e. $\widetilde{UL}[V,d] = UL[V,d]/I_L$. Let $\widetilde{C}[V,d]$ be the center of the algebra $\widetilde{UL}[V,d]$ and let $C[V,d] = \pi^{-1}(\widetilde{C}[V,d])$ be its pre-image in UL, where π is the natural projection $\pi : UL[V,d] \to \widetilde{UL}[V,d]$. We have the inclusion

$$\operatorname{Cen}(UL[V,d]) + I_L \subset C[V,d].$$

Let $S[V,d]$ be a $\mathfrak{sl}$-submodule in $C[V,d]$ which is the complement to the module $\operatorname{Cen}(UL[V,d]) + I_L$. Similarly let $C_0[V,d]$ and $S_0[V,d]$ be the corresponding modules, if we use the subalgebra $UL_0[V,d]$ instead of the whole algebra $UL[V,d]$.

LEMMA 3.7.2. *Let M denote the $\mathfrak{sl}$-module generated by the derivation $\mathrm{AD}\, x_1^2$. Then,*

$$\operatorname{Cen}_{\widetilde{UL}[V,d]}(M + I_L) = \{\delta \in UL[V,d] \mid [\delta, u] \in I_L \textit{ for all } u \in M\} =$$

$$\{\delta \in UL[V,d] \mid [\delta, u] = 0 \textit{ for all } u \in M\} = UL[V,d]^{(d-2)} + UL[V,d]^{(d-1)} + I_L.$$

PROOF. In order to prove the lemma it is enough to show that

$$\operatorname{Cen}_{\widetilde{UL}[V]}(M + I_L) = I_L.$$

Suppose that $\mathrm{AD}_f \in \operatorname{Cen}_{\widetilde{UL}[V]}(M + I_L)$ for some associative polynomial f. Let k be the smallest number such that f can be written as a sum of products of no more than k Lie polynomials (let us denote this number by $\mathrm{L}\deg f$). By Lemma 1.3.4, we have $[\mathrm{AD}_f, \mathrm{AD}_{x^2}] = \mathrm{AD}_h$, where

$$h = x^2 f - f x^2 + \mathrm{AD}_f(x^2) + \mathrm{AD}_{x^2}(f).$$

The assumption $\mathrm{AD}_f \in \mathrm{Cen}_{\widetilde{UL}[V,\infty]}(M + I_L)$ implies that h is a Lie polynomial. But it is easy to see that

$$\mathrm{L}\deg(x^2 f - f x^2) = k+1, \quad \mathrm{L}\deg \mathrm{AD}_f(x^2) = 2, \text{ and } \quad \mathrm{L}\deg \mathrm{AD}_{x^2}(f) = k.$$

This shows that if $k \geq 2$ then $\mathrm{L}\deg h = k+1 > 1$ which is a contradiction. Therefore, $k = 1$, i.e. f is a Lie polynomial and $\mathrm{AD}_f \in I_L$.

Suppose that ${}_y\Delta_g \in \mathrm{Cen}_{\widetilde{UL}[V,d]}(M + I_L)$ for some Lie polynomial g, which does not depend on y. Then, direct computation shows that

$$[\,{}_y\Delta_g, \mathrm{AD}_{x^2}] = \,{}_y\Delta_h$$

where $h = \mathrm{AD}_{x^2}(g) - [x,[x,g]]$ also does not depend on x. But this derivation can not be in I_L unless $h = 0$, which is impossible since $\deg g > 1$.

By Theorem 1.3.8 the Lie algebra $UL[V]$ is generated as an $\mathfrak{sl}$-module by the elements AD_f and ${}_y\Delta_g$, we have shown that

$$\mathrm{Cen}_{\widetilde{UL}[V,\infty]}(M + I_L) = I_L.$$

$\square$

Lemma 3.7.3. *Let $n \geq 3$ and let M denote the $\mathfrak{sl}$-module generated by the derivation ${}_{x_1}\Delta_{[x_2,x_3]}$. Then,*

$$\mathrm{Cen}_{\widetilde{UL}[V,d]}(M + I_L) = \{u \in UL[V,d] \mid [\delta, u] \in I_L \textit{ for all } u \in M\} =$$

$$\{u \in UL[V,d] \mid [\delta, u] = 0 \textit{ for all } u \in M\} = \mathrm{Cen}(UL[V,d]) = UL[V,d]^{(d-1)} + I_L.$$

Proof. Similar to the proof of the previous lemma. $\square$

Theorem 3.7.4. *The inclusion in Definition 3.7.1 is in fact an equality if $n \geq 3$, i.e. $S[V,d] = S_0[V,d] = 0$. In the case $n = 2$ we have that the two spaces are equal*

$$S[V,d] = S_0[V,d] = UL[V,d]^{(d-2)}/(I_L \cap UL[V,d]^{(d-2)}) = (UL^{(d-2)} + I_L)/I_L.$$

3.8. On Second Derivation Algebra

In this section we prove Theorems 3.8.13 and 3.8.16, which describe the derivation algebra of $\operatorname{Der}_0 L[V,d]$ and its algebra of restricted derivations In the next two sections we will use these theorems as a base for an induction to describe the tower of restricted derivations of the algebra $L[V,d]$

Let us consider the algebras $\operatorname{Der} U$ and $\operatorname{Der}_0 U$ of derivations of $U = \operatorname{Der}_0 L[V,d]$, where $d \geq 2$.

LEMMA 3.8.1. *The center of the Lie algebra U is*

$$\operatorname{Cen}(U) = (\operatorname{Cen}(UL[V,d]))^{\mathfrak{sl}} = (UL[V,d]^{(d-1)})^{\mathfrak{sl}}.$$

In particular $\operatorname{Cen}(U) = 0$ *if* $n \not| d-1$.

PROOF. The first equality holds because $U = \mathfrak{sl} + UL[V,d]$ and $\mathfrak{sl}$ is a simple Lie algebra. The second equality holds because $\operatorname{Cen}_{UL[V]}(\operatorname{ad}(V)) = 0$ (which follows from Lemma 1.3.4). □

The algebra of the inner derivations of U is isomorphic to $U/\operatorname{Cen}(U)$, so if we want to describe the derivation algebra of U we need only to describe the algebra of outer derivations. We start with any derivation D and we try to modify D using inner derivation to make D act trivially on as big subalgebra of U as possible.

DEFINITION 3.8.2. Let $\sim$ be the equivalence relation on $\operatorname{Der} U$ defined as follows: $a \sim b$ if and only if the derivation $a - b$ is an inner derivation of U.

LEMMA 3.8.3. *Any derivation is equivalent to one which kills* $\mathfrak{sl}$.

PROOF. The algebra $\mathfrak{sl}$ is simple and it is a standard fact in theory of semi-simple algebras that $H^1(\mathfrak{g}, K) = 0$ for any simple Lie algebra $\mathfrak{g}$ and any $\mathfrak{g}$-module K. □

Corollary 3.8.4. The Lie algebra $\mathfrak{sl}$ acts trivially on the algebra of outer derivations $\operatorname{Out} U$. Another way to prove that, is to use the fact that the inner derivations form an ideal, which contains $\mathfrak{sl}$. Therefore this ideal contains all non trivial $\mathfrak{sl}$ submodules of $\operatorname{Der} U$, i.e., the space $\operatorname{Out} U$ is a trivial $\mathfrak{sl}$ module. Therefore every element from this space acts the algebra $\mathfrak{sl}$.

First we will describe the space of derivations which act trivially on all inner derivations I_L. This is equivalent to saying that we are interested only in derivations which act trivially on the set $\operatorname{ad}(V)$.

Lemma 3.8.5. *Let D be a derivation of U such that $D(\operatorname{ad}(x)) \in \operatorname{Cen}(UL)$ for all $x \in V$. Then for any $d \in U$ and any $y \in V$, we have that $(Dd)(y) \in \operatorname{Cen}(L[V,d])$, i.e., $\operatorname{Im} D \in \operatorname{Cen}(UL[V,d])$. The last inclusion it immediately implies that D acts trivially on $[UL, UL]$.*

Proof. The algebra $L[V,d]$ is generated by V. Therefore, any such derivation d acts trivially on $\operatorname{ad}(L[V,d]) = I_L$. Finally we can apply Lemma 3.2.3 and obtain

$$\operatorname{Im} D \in \operatorname{Cen}_U(I_L) = \operatorname{Cen}(UL[V,d]).$$

□

Corollary 3.8.6. The set of derivations of U, which send $\operatorname{ad}(V)$ to $\operatorname{Cen}(UL[v,d])$ and act trivially on $\mathfrak{sl}$, can be embedded into the set

$$\operatorname{Hom}_{\mathfrak{sl}}\big(U_{\mathrm{gen},0}[V,d], \operatorname{Cen}(UL[V,d])\big) + \operatorname{Hom}_{\mathfrak{sl}}\big(U_{\mathrm{gen}}^{[0]}[V,d], \operatorname{Cen}(UL[V,d])^{\mathfrak{sl}}\big).$$

Every element in this set can be extended to a derivation of $\operatorname{Der}_0 L[V,d]$, by letting it act trivially on $\mathfrak{sl}$ and $[UL, UL]$. Therefore the above embedding is an isomorphism.

This is also an isomorphism of Lie algebras (the Lie algebra structure of $\mathrm{Hom}_{\mathfrak{sl}}(\dots)$ comes form the associative algebra structure on this space described in Remark 3.6.7).

Our next step is to see how the derivations of U can act on the space $\mathrm{ad}(V)$.

LEMMA 3.8.7. *For any derivation D of the algebra UL we have*

$$D(\mathrm{ad}(V)) \subset UL^{(1)} + C[V, d].$$

PROOF. Without loss of generality we may assume that D is homogeneous derivation of positive degree (if the degree is zero we have that $D(\mathrm{ad}(V)) \subset UL^{(1)}$). Since the space I_L is an ideal we have that $D([I_L, I_L]) \subset I_L$. Let $\delta \in UL$ be a derivation. Then for any $x \in V$ we have that

$$D([\mathrm{ad}(x), \delta]) = [D(\mathrm{ad}(x)), \delta] + [\mathrm{ad}(x), D(\delta)],$$

but $[\mathrm{ad}(x), D(\delta)] \in I_L$ and

$$D([\mathrm{ad}(x), \delta]) = D(\mathrm{ad}(\delta(x))) \in I_L.$$

Therefore, we have that $[D(\mathrm{ad}(x)), \delta] \in I_L$, which is equivalent to $D(\mathrm{ad}(x)) \in C[V, d]$. $\square$

LEMMA 3.8.8. *For any derivation D of the algebra UL, such that $D(\mathrm{ad}(V)) \in [I_L, I_L]$, there exists an element $f \in UL$ such that $(D - \mathrm{ad}(f))(\mathrm{ad}(V)) = 0$.*

PROOF. If D is a derivation such that $D(\mathrm{ad}(V)) \in [I_L, I_L]$, then D defines a map $d : V \to [L[V], L[V]]$ by $\mathrm{ad}(d(x)) = D(\mathrm{ad}(x))$. We can extend this map to a derivation f of $L[V]$. It is easy to see that $f \in UL$ and $(D - \mathrm{ad}(f))(\mathrm{ad}(V)) = 0$. $\square$

LEMMA 3.8.9. *Let $f \in \mathrm{Hom}_{\mathfrak{sl}}(V, S[V, d])$ be an $\mathfrak{sl}$-invariant linear map. Then there exists a derivation D of $\mathrm{Der}_0 L[V, d]$ such that $D(\mathrm{ad}(x)) = f(x)$ for all $x \in V$.*

Proof. The map f can be extended to a derivation from I_L to $UL[V, d]$ by the Leibnitz rule (because I_L is the 'free' Lie algebra generated by V). Let us denote this extension with D. Note that $D([I_L, I_L]) \subset I_L$.

Now we have to define $D(\delta)$ for any $\delta \in UL$. If such an extension exists we would have

$$D(\mathrm{ad}(\delta(x))) = D([\mathrm{ad}(x), \delta]) = [D(\mathrm{ad}(x)), \delta] + [\mathrm{ad}(x), D(\delta)].$$

Therefore

$$\mathrm{ad}\, D(\delta)(x) = [f(x), \delta] - D(\mathrm{ad}(\delta(x))).$$

By the definition of the module S we have that $[f(x), \delta] \in I_L$. Also we have that $D(\mathrm{ad}(\delta(x))) \in I_L$, because $\mathrm{ad}(\delta(x)) \in [I_L, I_L]$. Therefore, the right side is an element in I_L, which allows us to define $D(\delta)(x)$ by the above equality. Thus, we have defined $D(\delta)$. It is easy to check that D is a derivation of UL. Finally setting $D(\mathfrak{sl}) = 0$, defines D as a derivation of $\mathrm{Der}_0 L[V, d]$. □

Remark 3.8.10. Note that the statement of this lemma is trivial unless $n = 2$ and $2|d$, because if $n > 2$ by, Corollary 3.7.4, $S = 0$ and if $n = 2$ and $2 \not| d$, then $\mathrm{Hom}_{\mathfrak{sl}}(V, S) = 0$. Also using the fact that the space S is so small there is no need to check that the map defined in the proof above is a derivation because it sends 'almost' every element to 0.

Now we can describe the derivation algebra of U.

Theorem 3.8.11. *If* $\mathrm{Cen}(U) = 0$ *then the derivation algebra of* U *can be written as*

$$\mathrm{Der}\, U = \mathfrak{gl} + UL + \sum_\lambda W_0^\lambda + T,$$

where

$$W_0^\lambda[V,d] = \mathrm{Hom}_{\mathfrak{sl}}(U^\lambda_{\mathrm{gen},0}, CUL_0^\lambda[V,d]),$$

for $\lambda \neq [1]$ *and*

$$W_0^{[1]}[V,d] = \mathrm{Hom}_{\mathfrak{sl}}(U^{[1]}_{\mathrm{gen},0}, CUL_0^{[1]}[V,d])/(UL^{(d-2)})^{\mathfrak{sl}}.$$

Finally we have that

$$T[V,d] \simeq \mathrm{Hom}_{\mathfrak{sl}}\left(\mathrm{ad}(V), S[V,d]\right) =$$
$$\begin{cases} 0 & \textit{if } n \geq 3 \textit{ or } d \leq 3 \\ \mathrm{Hom}_{\mathfrak{sl}}(\mathrm{ad}(V), UL[V,d]^{(d-2)} + I_L/I_L) & \textit{if } n = 2 \textit{ and } d \geq 4 \end{cases}.$$

The Lie algebra structure on this space is given by the natural action of $\mathfrak{gl}$ *on* UL*, and its trivial action on* W_0^λ *and* T*; also* W_0^λ *and* T *act naturally* UL *in particular they act trivially on* $[UL, UL]$.

Remark 3.8.12. The space T is trivial, because if $n \geq 3$ then S is trivial. In the case $n = 2$, the condition $\mathrm{Cen}(U) = 0$ implies that $2|d$ therefore the space $\mathrm{Hom}_{\mathfrak{sl}}(\mathrm{ad}(V), UL^{(d-2)})$ is trivial because the source has degree 1; the target has even degree and all $\mathfrak{sl}$ invariant maps are of even degree.

Proof. The algebra of inner derivations of the U is the part $\mathfrak{sl} + UL$, because the center of U is trivial.

By Lemma 3.8.5 and Corollary 3.8.6 we have that the space W of outer derivations (which acts 'almost' trivially on $\mathrm{ad}(V)$) is isomorphic to

$$\mathrm{Hom}_{\mathfrak{sl}}\left(U_{\mathrm{gen},0}[V,d], \mathrm{Cen}(UL[V,d])\right),$$

since the part coming from $U^{[0]}_{\mathrm{gen}}$ is trivial because $\mathrm{Cen}(U) = 0$. Lemmas 3.8.7, 3.8.8 and 3.8.9 describe the set of derivations which act 'non-trivially' in $\mathrm{ad}(V)$ modulo the inner derivations. We need to factor out some part of the space $W_0^{[1]}$ because

$(UL^{(d-2)})^{\mathfrak{sl}}$ is part of the algebra of inner derivations and also part of $\mathrm{Hom}_{\mathfrak{sl}}(V, \mathrm{Cen}(UL))$. □

Now we can describe the second algebra in the restricted derivation tower of $L[V,d]$.

THEOREM 3.8.13. *The algebra* $\mathrm{Der}_0^2 L[V,d]$ *can be written as*

$$\mathrm{Der}_0^2 L[V,d] = \mathfrak{sl} + UL[V,d] + \sum_{\lambda} R_0^{\lambda}[V,d] + T$$

provided that $\mathrm{Cen}(\mathrm{Der}_0 L[V,d]) = 0$. *Here we have denoted*

$$R_0^{\lambda}[V,d] = \mathrm{Hom}^0_{\mathfrak{sl}}(U^{\lambda}_{\mathrm{gen},0}, CUL_0^{\lambda}[V,d]),$$

for $\lambda \neq [1]$ *and*

$$R_0^{[1]}[V,d] = \mathrm{Hom}^0_{\mathfrak{sl}}(U^{[1]}_{\mathrm{gen},0}, CUL_0^{[1]}[V,d])/(UL^{(d-2)})^{\mathfrak{sl}}.$$

PROOF. Theorem 3.8.11 gives a description of the full derivation algebra. Here we are interested only in restricted derivations, which correspond to putting Hom^0 instead of Hom. □

REMARK 3.8.14. The difference between the reductive part of $\mathrm{Der}_0^2 L$ and $\mathrm{Der}_0 L$ is just a sum of copies of $\mathfrak{sl}_l$, for some l-es. It acts faithfully on the nilpotent part and all nontrivial modules for some $\mathfrak{sl}_l$ are isomorphic to the standard or to its dual. Therefore the reductive part of $\mathrm{Der}_0^2 L$ is a semi simple Lie algebra which acts faithfully on the nilpotent part.

THEOREM 3.8.15. *The restricted derivation tower of the free nilpotent Lie algebra of class* $d \leq n$ *terminates at the second level, i.e.* $\mathrm{Der}_0^2 L[V,d] = \mathrm{Der}_0^1 L[V,d]$.

PROOF. Theorem 3.8.13 describes the algebra $\operatorname{Der}_0^2 L[V,d]$. In this case we have

$$R_0^\lambda = \operatorname{Hom}(U_{\mathrm{gen},0}^\lambda, CUL_0^\lambda[V,d])$$

but this space is trivial unless $|\lambda| = d-1$, because we do not have $\mathfrak{sl}$ invariant maps except when n divides the difference of the degree. If $|\lambda| = d-1$, then by Theorem 1.4.10, $U_{\mathrm{gen},0}^\lambda$ is trivial unless for $\lambda = [d-1]$. In the last case we have that both $U_{\mathrm{gen},0}^\lambda$ and $CUL_0^\lambda[V,d]$ coincide and are simple $\mathfrak{sl}$ modules therefore $\operatorname{Hom}_{\mathfrak{sl}}$ is one dimensional, but $\operatorname{Hom}_{\mathfrak{sl}}^0$ is trivial. □

In the case when the center of U is not trivial the situation is slightly more complicated:

THEOREM 3.8.16. *If* $\operatorname{Cen}(U) \neq 0$ *then the derivation algebra of* U *decomposes*

$$\operatorname{Der} U = \mathfrak{gl} + UL/\operatorname{Cen}(U) + \sum_\lambda W_0^\lambda + T$$

and the algebra of restricted derivations is

$$\operatorname{Der}_0^2 L[V,d] = \mathfrak{sl} + UL[V,d]/\operatorname{Cen}(\operatorname{Der}_0 L[V,d]) + \sum_\lambda R_0^\lambda[V,d] + T$$

where W_0^λ *and* R_0^λ *for* $\lambda \neq [0]$ *are defined as in Theorems 3.8.11 and 3.8.13, and*

$$W_0^{[0]} = \operatorname{Hom}_{\mathfrak{sl}}(U_{\mathrm{gen}}^{[0]}, \operatorname{Cen} U) \quad \textit{and} \quad R_0^{[0]} = \operatorname{Hom}_{\mathfrak{sl}}^0(U_{\mathrm{gen}}^{[0]}, \operatorname{Cen} U).$$

Also note that $W_0^{[0]}$ *and* $R_0^{[0]}$ *split as a direct summands of* $\operatorname{Der}\operatorname{Der}_0 L[V,d]$ *and* $\operatorname{Der}_0^2 L[V,d]$ *respectively.*

PROOF. The proof of this theorem is similar to the one of Theorem 3.8.11. The only thing that needs to be checked is that $W_0^{[0]}$ acts trivially on the rest of the algebra $\operatorname{Der} U$. This is true because $W_0^{[0]}(U) \subset \operatorname{Cen}(U)$ and everything else acts trivially on the center of U. Therefore it splits as a direct summand. □

LEMMA 3.8.17. *The centralizer of the ideal* $\mathrm{ad}(UL_0)$ *in* $\mathrm{Der}_0^2 L[V,d]$ *is isomorphic to* $W_0^{[0]}[V,d] + \mathrm{Cen}(UL[V,d])/\mathrm{Cen}(\mathrm{Der}_0 L)$.

PROOF. The centralizer of $\mathrm{ad}(UL_0)$ in $UL[V,d]/\mathrm{Cen}(\mathrm{Der}_0 L)$ is

$$\mathrm{Cen}(UL[V,d])/\mathrm{Cen}(\mathrm{Der}_0 L),$$

and for $\lambda \neq [0]$, the space W_0^λ acts faithfully on $U_{\mathrm{gen}}^\lambda \subset UL_0$. Therefore the centralizer of $\mathrm{ad}(UL_0)$ in $\mathrm{Der}_0^2 L[V,d]$ is contained in

$$W_0^{[0]}[V,d] + \mathrm{Cen}(UL[V,d])/\mathrm{Cen}(\mathrm{Der}_0 L) + T[V,d].$$

It is easy to see that it is only $W_0^{[0]}[V,d] + \mathrm{Cen}(UL[V,d])/\mathrm{Cen}(\mathrm{Der}_0 L)$, because T act faithfully on $\mathrm{ad}(V) \subset UL_0$. $\square$

LEMMA 3.8.18. *The center of the algebra* $\mathrm{Der}_0^2 L[V,d]$ *coincides with the center of* $R_0^{[0]}$.

PROOF. By the previous lemma we have that the center of $\mathrm{Der}_0^2 L[V,d]$ lies inside $R_0^{[0]}$, because $\mathfrak{sl}$ acts nontrivially on $\mathrm{Cen}(UL)/\mathrm{Cen}(U)$. Using the algebra $\mathrm{Der}_0^2 L[V,d]$ splits as a direct sum, it is easy to see that its center is exactly $\mathrm{Cen}\, R_0^{[0]}$. $\square$

REMARK 3.8.19. The analog of the Remark 3.8.14 holds also in the case when the center of U is not trivial.

3.9. Derivation tower of free nilpotent Lie algebras. Case with trivial center

The description of the second derivation algebra can be generalized to a description of all algebras in the restricted derivation tower. As we saw in the previous section

there is a big difference in the structure of the algebra $\operatorname{Der}_0^2 L[V,d]$ depending on the triviality of the center of $\operatorname{Der}_0 L[V,d]$.

Let us first consider the case when $n \not| d-1$ and the algebra $\operatorname{Der}_0 L[V,d]$ has a trivial center. It is easy to see that this implies that all algebras in the restricted derivation tower have trivial centers, and all the maps $\operatorname{Der}_0^k L[V,d] \to \operatorname{Der}_0^{k+1} L[V,d]$ are embeddings.

The next theorem gives the description of all algebras in the restricted derivation tower and its proof is by induction where Theorem 3.8.11 serves as a base case.

THEOREM 3.9.1. *The derivation algebra of* $\operatorname{Der}_0^k L[V,d]$*, for* $k \geq 2$ *is*

$$\operatorname{Der}\operatorname{Der}_0^k L[V,d] = \operatorname{Der}_0^k L[V,d] + \sum_\lambda W_{k-1}^\lambda,$$

where

$$W_k^\lambda = \operatorname{Hom}_{\mathfrak{sl}}(U_{\mathrm{gen},k}^\lambda, CUL_k^\lambda).$$

The algebra of restricted derivations is obtained by replacing W_{k-1}^λ *with its subspace* R_{k-1}^λ*, consisting of maps with zero trace on* $CU_{\mathrm{gen},k-1}^\lambda$*.*

PROOF. Proof is by induction on k. Suppose that the theorem is true for some k (in case $k=2$ we use Theorem 3.8.13). In order to make the induction step we have to describe the algebras $\operatorname{Der}\operatorname{Der}_0^k L[V,d]$ and $\operatorname{Der}_0 \operatorname{Der}_0^k L[V,d]$.

Let us describe the derivation algebra $\operatorname{Der}\operatorname{Der}_0^k L[V,d]$. First we want to see how the outer derivations can act on the algebra $\operatorname{Der}_0^k L[V,d]$.

LEMMA 3.9.2. *Let* $D \in \operatorname{Der}\operatorname{Der}_0^k L[V,d]$ *be a derivation of* $\operatorname{Der}_0^k L[V,d]$ *such that* $D(\operatorname{ad}(V)) = \operatorname{Cen}(UL)$ *and* $D(\mathfrak{sl}) = 0$*. Then* $D([UL^+, UL^+]) = 0$*.*

PROOF. By Lemma 3.2.5, $D(f)$ commutes with $\mathrm{ad}(V)$ for any $f \in UL$. Note that the space of elements in $\mathrm{Der}_0^k L[V,d]$ which commute with $\mathrm{ad}(V)$ is

$$\mathrm{Cen}_{\mathrm{Der}_0^k L[V,d]}(\mathrm{ad}(V)) = \mathrm{Cen}(UL) + \sum_{i=0}^{k-2} \sum_{\lambda} R_i^{\lambda}[V,d].$$

All spaces $R_i^{\lambda}[V,d]$ are trivial $\mathfrak{sl}$-modules. Therefore $D(U_\mu) \subset \mathrm{Cen}(UL)$ for all simple modules U_μ corresponding to the partition $\mu \neq [0]$. The space $\mathrm{Cen}(UL)$ is an abelian Lie algebra. Therefore D acts trivially on the commutator algebra of the algebra generated by UL^+. □

LEMMA 3.9.3. *Let $D \in \mathrm{Der}\,\mathrm{Der}_0^k L[V,d]$ be a derivation of $\mathrm{Der}_0^k L[V,d]$ such that $D(\mathrm{ad}(V)) = 0$ and $D(\mathfrak{sl}) = 0$. Then*

$$D(x) \in \mathrm{Cen}(UL) + \sum_{i=0}^{k-2} \sum_{\lambda} P_i^{\lambda}[V,d],$$

for all $x \in UL[V,d]$, where P_i^{λ} is the subspace of R_i^{λ} consisting of all maps which are trivial on $CU^{\lambda}_{\mathrm{gen},i}$.

PROOF. Assume the contrary — then there exists $g \in \mathrm{Cen}(UL)_\lambda$ such that $D(x)(g) \neq 0$. Therefore, $[D(x),g] = D(x)(g) \neq 0$. But this is impossible because $D(g) = 0$ and $[x,g] = 0$. □

LEMMA 3.9.4. *Let $D \in \mathrm{Der}\,\mathrm{Der}_0^k L[V,d]$ be a derivation of $\mathrm{Der}_0^k L[V,d]$ such that $D(\mathrm{ad}(V)) = 0$ and $D(\mathfrak{sl}) = 0$. Then $D(UL_k[V,d]) = 0$.*

PROOF. Let u be an element in $u \in UL^{\lambda}_{(i)}$ for some $i \geq k$, which can be written as $u = [\tilde{u}, u_0]$, where $\tilde{u} \in UL^{\lambda}_{(i-1)}$ and $u_0 \in U_{[0]}$. We have

$$D(u) = [D(\tilde{u}), u_0] + [\tilde{u}, D(u_0)].$$

Using $\mathfrak{sl}$ invariance we have $D(\tilde{u}) \in \mathrm{Cen}(UL)$ and, therefore $D(\tilde{u})$ commutes with u_0. Also $D(u_0) \in \sum_{j\le k,\lambda} R_j^\lambda$ and by the induction hypothesis acts trivially on $UL^0_{k-1}[V,d]$. Since $\tilde{u} \in UL_{k-1}[V,d]$, we have that $D(u) = 0$.

Finally, we notice that any element in $U^\lambda_{(i)}$ is a sum of the elements considered above, which completes the proof. $\square$

LEMMA 3.9.5. *Let $D \in \mathrm{Der}\,\mathrm{Der}^k_0 L[V,d]$ be a derivation of $\mathrm{Der}^k_0 L[V,d]$ such that $D(\mathrm{ad}(V)) = 0$ and $D(\mathfrak{sl}) = 0$. Then $D(UL^\lambda_{k-1}) \subset CU^\lambda_{k-1}$, for any partition $\lambda \neq [0]$.*

PROOF. Assume the contrary, ie., that $D(x) \in CUL^\lambda_{i-1} \setminus CUL^\lambda_i$, for some $x \in UL^\lambda_{k-1}$ and some $i < k$. Then there exists some $f \in R^\lambda_{i-1}$ such that $[f, D(x)] \in CUL^\lambda \setminus 0$. By the choice of f we have $[f,x] = 0$, which implies that $[D(f),x] \in CUL^\lambda \setminus 0$, but this is impossible because all R^μ_j act trivially on x. This is a contradiction which finishes the proof of the lemma. $\square$

LEMMA 3.9.6. *Let $f \in \sum_\lambda \mathrm{Hom}_{\mathfrak{sl}}\left(U^\lambda_{\mathrm{gen},k-1}, CUL^\lambda_{k-1}\right)$ be a $\mathfrak{sl}$ invariant linear map. Then there exists a derivation $D \in \mathrm{Der}\,\mathrm{Der}^k_0 L[V,d]$ such that the restriction of D to $\sum U^\lambda_{\mathrm{gen},k-1}$ coincides with f.*

PROOF. Let us construct a derivation D which acts trivially on $\mathfrak{sl}$, UL_k, T and $R^\lambda_i[V,d]$ for any partition λ and all $i \le k$. This allows us to lift the map f to a map D form UL_{k-1} to CUL^+_{k-1}. Let us extend D to an $\mathfrak{sl}$ invariant map from UL_0 to CLU^+_{k-1}. In order to define D as a derivation of $\mathrm{Der}^k_0 L[v,d]$ we only need to say how it acts on $UL^{[0]}$ and verify that it satisfies the Leibnitz rule.

Now we define $D : UL^[0] \to \sum R^\mu_j$ by

$$D(u)(v) = D[u,v],$$

for any $u \in UL^{[0]}$ and $v \in UL_0$.

In order to verify that D is well defined, we need to verify that for any $u \in UL_k \cap UL^{[0]}$, we have that $D(u) = 0$. Any such u has to lie in $[UL^+, UL^+]$ and therefore $[u, v] \in [UL^+, UL^+] \subset UL_k$, This shows that $D(u) = 0$, which gives that D is well defined.

Finally we have to check that D satisfies the Leibnitz rule, which can be verified using the construction of D. $\square$

REMARK 3.9.7. Lemmas 3.8.7 and 3.8.8 also apply in this case, with one exception — since $UL[V, d]$ is no longer closed under derivations, we have to work with $UL_0[V, d]$, which is still closed. Therefore, we need to substitute $S[V, d]$ with $S_0[V, d]$, but Corollary 3.7.4 gives that these two spaces are the same. Thus, for any $D \in \operatorname{Der} \operatorname{Der}_0^k L[V, d]$ there exists an inner derivation $\operatorname{ad}(d)$ and a scalar c such that $D - d$ acts on $\operatorname{ad}(V)$ as a multiplication by c.

LEMMA 3.9.8. *Let $D \in \operatorname{Der} \operatorname{Der}_0^k L[V, d]$ be a derivation of $\operatorname{Der}_0^k L[V, d]$ such that $D(UL_0) = 0$ and $D(\mathfrak{sl}) = 0$. Then $D = 0$.*

PROOF. The centralizer of the space UL_0 in $\operatorname{Der}_0^k L[V, d]$ is $\operatorname{Cen} UL$. Lemma 3.2.5 gives us that $\operatorname{Im} D \subset \operatorname{Cen} UL$. The last space does not have any $\mathfrak{sl}$ invariant elements. Therefore D acts trivially on any trivial $\mathfrak{sl}$ module in $\operatorname{Der}_0^k L[V, d]$. By construction the factor algebra $\operatorname{Der}_0^k L[V, d]/UL_0$ is $\mathfrak{sl}$ invariant, therefore D is trivial. $\square$

We can combine the results of the previous lemmas to obtain a description of the derivation algebra of $\operatorname{Der}_0^k L[V, d]$.

THEOREM 3.9.9. *The derivation algebra of $\operatorname{Der}_0^k L[V, d]$ can be written as*

$$\operatorname{Der} \operatorname{Der}_0^k L[V, d] = \operatorname{Der}_0^k L[V, d] + \mathbb{C}.d + \textstyle\sum_\lambda W_{k-1}^\lambda[V, d],$$

where

$$W^{\lambda}_{k-1}[V,d] \simeq \mathrm{Hom}_{\mathfrak{sl}}\left(\mathrm{Hom}_{\mathfrak{sl}}(U^{\lambda}_{\mathrm{gen},k-1}, CUL^{\lambda}_{k-1})\right).$$

REMARK 3.9.10. Note that above we showed that the set of derivations from $\mathfrak{sl} + UL_0$ to $\mathrm{Der}_0^k L[V,d]$ is isomorphic to $\mathrm{Der}\,\mathrm{Der}_0^k L[V,d]$. We are going to use this fact in chapter 4.

Now we can finish the proof of Theorem 3.9.1. We only have to notice that in order to pass from $\mathrm{Der}\,\mathrm{Der}_0^k L[V,d]$ to $\mathrm{Der}_0^{k+1} L[V,d]$, we need to substitute all Hom operators with Hom^0's. After applying this to $W^{\lambda}_{k-1}[V,d]$, we get $R^{\lambda}_{k-1}[V,d]$, which completes the proof of the theorem. □

The next corollaries are an immediate consequence of the above Theorem and Theorem 3.6.4.

COROLLARY 3.9.11. If $n \not| d-1$, then the restricted derivation tower of $L[V,d]$ stabilizes after at most $3 + d/h(n)$ steps.

COROLLARY 3.9.12. If $n < d < n(n-1)$ (or if Conjecture 1.5.2 holds) then the restricted derivation tower of $L[V,d]$ stabilizes at the third level if $n \not| d-1$.

REMARK 3.9.13. The difference between the reductive part of $\mathrm{Der}_0^{k+1} L$ and $\mathrm{Der}_k L$ is just a sum of copies of $\mathfrak{sl}_l$, for some l-es. It acts faithfully on the nilpotent part and all nontrivial modules for some $\mathfrak{sl}_l$ are isomorphic to the standard or to its dual. Therefore the reductive part of $\mathrm{Der}_0^k L$ is a semi simple Lie algebra which decomposes as a sum of copies of $\mathfrak{sl}_l$ for some l-es. It also acts faithfully on the nilpotent part and almost all nontrivial modules which appear in are either isomorphic to the standard one or its dual.

3.10. Derivation tower of free nilpotent Lie algebras. Case with non trivial center

Now let us consider the case $n|d-1$. This case is similar to the case when the center is trivial, but there are two important things to consider. First, the centers of some algebras in the derivation tower are not trivial, and the map between two algebras in the tower is not always an embedding. Also all algebras in the tower split as a direct sum of two subalgebras.

THEOREM 3.10.1. *For $k>2$, the k-th algebra in the restricted derivation tower of $L[V,d]$ can be written as a direct sum $\operatorname{Der}_0^k L[V,d] = DL^k[V,d] \oplus \tilde{R}_k^{[0]}[V,d]$.*

The algebras $DL^k[V,d]$ and $\tilde{R}_k^{[0]}[V,d]$ can be constructed recursively by

$$DL^{k+1}[V,d] = DL^k[V,d] + \sum_{\lambda \neq [0]} R_{k-1}^{\lambda}[V,d]$$

where the spaces R_{k-1}^{λ} of outer derivations for $\lambda \neq [0]$ are as in Theorem 3.9.1, and

$$\tilde{R}_k^{[0]} = \operatorname{Der}_0 \tilde{R}_{k-1}^{[0]} + \operatorname{Hom}\left(DL^{k-1}/[DL^{k-1}, DL^{k-1}], \operatorname{Cen}(\tilde{R}_{k-1}^{[0]})\right).$$

PROOF. As the proof of Theorem 3.9.1, this theorem is proved on induction. The base case $k=2$ is given by Theorem 3.8.16.

The proof is essentially the same as the proof in the case with trivial center, the main differences are:

a) In order to prove Lemma 3.9.2 we need to use the fact that $\tilde{R}_i^{[0]}[V,d]$ commutes with $UL_0[V,d]$, which is part in the induction hypothesis.

b) Lemma 3.9.8 does not hold in this case, but it can be generalized to

LEMMA 3.10.2. *Let D be a derivation of $\operatorname{Der}_0^k L[V,d]$ such that $D(UL_0)=0$ and $D(\mathfrak{sl})=0$, then $\operatorname{Im} D \subset \tilde{R}_k^{[0]}$.*

PROOF. As in Lemma 3.9.8 we want to apply Lemma 3.2.1, therefore we need to describe the centralizers of UL_0 in $\operatorname{Der}_0^k L[V,d]$. We have that

$$\operatorname{Cen}_{\operatorname{Der}_0^k L[V,d]}(UL_0) = \tilde{R}_k^{[0]} + \operatorname{Cen}(UL)/\operatorname{Cen}(UL)^{\mathfrak{sl}},$$

as before $\operatorname{Cen}(UL)/\operatorname{Cen}(UL)^{\mathfrak{sl}}$ does not contain $\mathfrak{sl}$ invariant vectors, and $\mathfrak{sl}$ acts trivially on $\operatorname{Der}_0^k L[V,d]/UL_0$. Therefore the image of D has to lie in the subalgebra $\tilde{R}_k^{[0]}$. □

LEMMA 3.10.3. *The space of derivations from* $\operatorname{Der}_0^k L[V,d]$ *to* $\tilde{R}_k^{[0]}$ *is*

$$\tilde{R}_{k+1}^{[0]} = \operatorname{Der} \tilde{R}_k^{[0]} + \operatorname{Hom}\left(DL^k/[DL^k, DL^k], \operatorname{Cen}(\tilde{R}_k^{[0]})\right).$$

PROOF. Let L be a Lie algebra such that $L = L_1 \oplus L_2$, where L_1 and L_2 are ideals. Let D be a derivation of L with image in L_1; using the fact that L_1 and L_2 commute, we can see that D kills $[L_2, L_2]$, and $D(L_2) \subset \operatorname{Cen} L_1$. This gives that the space of derivations form L to L_1 is isomorphic to

$$\operatorname{Hom}(L_2/[L_2, L_2], \operatorname{Cen}(L_1)) + \operatorname{Der} L_1.$$

In order to prove the lemma we need to apply the above argument to $L = \operatorname{Der}_0^k L[V,d]$, $L_1 = \tilde{R}_k^{[0]}$ and $L_2 = DL^k$. □

As in Theorem 3.9.9, we can describe the derivation algebra of $\operatorname{Der}_0^k L[V,d]$.

THEOREM 3.10.4. *The derivation algebra of* $\operatorname{Der}_0^k L[V,d]$ *can be written as*

$$\operatorname{Der}\operatorname{Der}_0^k L[V,d] = \left(DL^k[V,d] + \mathbb{C}.d + \textstyle\sum_{\lambda\neq[0]} \widetilde{W}_{k-1}^{\lambda}[V,d]\right) \oplus \tilde{W}_{k+1}^{[0]}[V,d],$$

where $\tilde{W}_{k+1,\lambda}[V,d]$ *is defined in Theorem 3.9.9 for* $\lambda \neq [0]$*, and in Lemma 3.10.3 for* $\lambda = [0]$*.*

To finish the proof of Theorem 3.10.1, we only need to substitute all Hom and Der operators with Hom^0's and Der_0's, respectively. $\square$

The sequence DL^k stabilizes after finitely many steps because the sequence UL_k stabilizes. In order to show that the restricted derivation tower of $L[V,d]$ stabilizes in this case we need to show that the sequence $\tilde{R}_k^{[0]}$ stabilizes.

LEMMA 3.10.5. *If $\tilde{R}_2^{[0]} \neq 0$ then for all $k \geq 2$*

$$\dim DL^k/[DL^k, DL^k] \geq 2.$$

PROOF. If $n \geq 3$ is follows from Theorem 1.4.11. Because in $\mathrm{Cen}\, UL$ there are many $\mathfrak{sl}$ (more than 2) modules of type $[n+s, s^{n-1}]$ for $s = (d-1)/n - 1$, because in the free Lie algebra $L[V]$ there are many modules of type $[n+s+1, s^{n-1}]$ (here we are using the $d > h(n) \geq n(n-1)$). The same argument works in the case $n = 2$ if $d \geq 7$. Therefore the only case which is left is $n = 2$ and $d = 5$. In this case the direct computation shows that $\dim DL^k/[DL^k, DL^k] = 2$ for all $k \geq 2$. $\square$

LEMMA 3.10.6. *The sequence*

$$\tilde{R}_2^{[0]} \to \tilde{R}_3^{[0]} \to \tilde{R}_4^{[0]} \to \tilde{R}_5^{[0]} \to \cdots$$

stabilizes at most at the sixth level, i.e., $\tilde{R}_6^{[0]} = \tilde{R}_7^{[0]}$.

PROOF. The behavior of this sequence is determined by the dimensions of the spaces $U_{\mathrm{gen}}^{[0]}$, $\mathrm{Cen}(UL)^{\mathfrak{sl}}$ and $\mathrm{Cen}(UL)^{\mathfrak{sl}} \cap [UL, UL]$. Let us define p, q, r as follows

$$p = \dim(UL/(UL^+ + \mathrm{Cen}(UL) + [UL, UL]), \quad q = \dim(\mathrm{Cen}(UL)^{\mathfrak{sl}} \cap [UL, UL]) \text{ and}$$

$$r = \dim(\mathrm{Cen}(UL)^{\mathfrak{sl}} / \mathrm{Cen}(UL)^{\mathfrak{sl}} \cap [UL, UL]).$$

Let also denote $l_i = \dim(DL^i/[DL^i, DL^i])$, by Lemma 3.10.5 we have that $l_i \geq 2$ for all i unless $\tilde{R}_2^{[0]} = 0$. Let us consider the different possibilities for this dimensions:

(1) $p=r=0$ or $r=q=0$. In this case all spaces $\tilde{R}_i^{[0]}$, for $i\geq 2$ are trivial and the sequence stabilizes.

(2) $r=0$, $p=q=1$. In this case we case we have $\tilde{R}_2^{[0]}=\mathbb{R}$ is abelian, this implies that $\tilde{R}_3^{[0]}=\mathbb{R}^{l_2}$ is also abelian and $\tilde{R}_4^{[0]}=\mathfrak{sl}_{l_2}+\mathbb{R}^{l_2l_3}$ and finally $\tilde{R}_i^{[0]}=\mathfrak{sl}_{l_2}+\mathbb{R}^{l_2l_3}+\mathfrak{sl}_{l_3}$ for $i\geq 5$.

(3) $r=0$ and $p.q>1$ In this case we case we have $\tilde{R}_2^{[0]}=\mathbb{R}^{pq}$ is abelian, therefore $\tilde{R}_3^{[0]}=\mathfrak{sl}_{pq}+\mathbb{R}^{pql_2}$ and finally $\tilde{R}_i^{[0]}=\mathfrak{sl}_{pq}+\mathbb{R}^{pql_2}+\mathfrak{sl}_{l_2}$ for $i\geq 4$.

(4) $r=1$ and $p=q=0$. In this case all spaces $\tilde{R}_i^{[0]}$, for $i\geq 2$ are trivial and the sequence stabilizes.

(5) $r=1$ and $p=0$ or $q=0$(say $q=0$). $\tilde{R}_2^{[0]}=\mathbb{R}^p$ is abelian is in cases 2 and 3, the sequence $\tilde{R}_i^{[0]}$ stabilizes after 5 or 6 steps depending on whether $p=1$ or not.

(6) $r=1$ and $p=q=1$. In this case $\tilde{R}_2^{[0]}$ is three dimensional nilpotent of class 2 Lie algebra. Therefore we have $\tilde{R}_3^{[0]}=\mathfrak{sl}_2+\mathbb{R}^2\oplus\mathbb{R}^{l_2}$. Similarly to case 3 the sequences $\tilde{R}_i^{[0]}$ stabilizes after 5 steps.

(7) $r=1$ and $p=1$ or $q=1$. $\tilde{R}_2^{[0]}=\mathbb{R}+\mathbb{R}^p+\mathbb{R}^p$, where one copy of $\mathbb{R}^p$ lies in the center and commutation with elements in $\mathbb{R}$ map one copy of $\mathbb{R}^p$ to the other. After that we have $\tilde{R}_3^{[0]}=\mathbb{R}+\mathbb{R}^p+\mathfrak{sl}_p+\mathbb{R}^{pl_2}$, $\tilde{R}_4^{[0]}=\mathbb{R}^p+\mathfrak{sl}_p+\mathbb{R}^{pl_2}+\mathfrak{sl}_{l_2}+\mathbb{R}^{l_3}$ and $\tilde{R}_5^{[0]}=\mathbb{R}^p+\mathfrak{sl}_p+\mathbb{R}^{pl_2}+\mathfrak{sl}_{l_2}+\mathfrak{sl}_{l_3}+\mathbb{R}^{l_3l_4}$ and finally

$$\tilde{R}_6^{[0]}=\mathbb{R}^p+\mathfrak{sl}_p+\mathbb{R}^{pl_2}+\mathfrak{sl}_{l_2}+\mathfrak{sl}_{l_3}+\mathbb{R}^{l_3l_4}+\mathfrak{sl}_{l_4}$$

for all $i\geq 5$

(8) $r=1$ and $p,q>1$. We have that $\tilde{R}_2^{[0]}=\mathbb{R}^p+\mathbb{R}^q+\mathbb{R}^{pq}$, where $\mathbb{R}^{pq}$ is in the center and the commutator of $\mathbb{R}^p$ and $\mathbb{R}^q$ is $\mathbb{R}^{pq}$. By the construction of $\tilde{R}_i^{[0]}$

we have

$$\tilde{R}_3^{[0]} = \mathfrak{sl}_p + \mathfrak{sl}_q + \mathbb{R}^p + \mathbb{R}^q + \mathbb{R}^{pql_2}$$

and for $i \geq 4$ we have

$$\tilde{R}_i^{[0]} = \mathfrak{sl}_p + \mathfrak{sl}_q + \mathbb{R}^p + \mathbb{R}^q + \mathbb{R}^{pql_2} + \mathfrak{sl}_{l_2}.$$

(9) $r \geq 2$ and $p = q = 0$. $\tilde{R}_i^{[0]} = \mathfrak{sl}_r$ for all $i \geq 2$.

(10) $r \geq 2$ and $p = 0$ or $q = 0$. $\tilde{R}_2^{[0]} = \mathfrak{sl}_r + \mathbb{R}^{r(p+q)}$. If $p + q = 1$ then all $\tilde{R}_i^{[0]}$ are equal to $\tilde{R}_2^{[0]}$, otherwise we have $\tilde{R}_i^{[0]} = \mathfrak{sl}_r + \mathbb{R}^{r(p+q)} + \mathfrak{sl}_{p+q}$ for all $i \geq 3$.

(11) $r \geq 2$ and $p = q = 1$. $\tilde{R}_2^{[0]} = \mathfrak{sl}_r + \mathbb{R}^r + \mathbb{R}^r + \mathbb{R}$, where $\mathbb{R}$ is in the center. $\tilde{R}_3^{[0]} = \mathfrak{sl}_r + \mathbb{R}^r + \mathbb{R}^r + \mathbb{R}^{l_2}$, and $\tilde{R}_4^{[0]} = \mathfrak{sl}_r + \mathbb{R}^r + \mathbb{R}^r + \mathfrak{sl}_{l_2} + \mathbb{R}^{l_2 l_3}$, and finally

$$\tilde{R}_i^{[0]} = \mathfrak{sl}_r + \mathbb{R}^r + \mathbb{R}^r + \mathfrak{sl}_{l_2} + \mathbb{R}^{l_2 l_3} + \mathfrak{sl}_{l_3},$$

for all $i \leq 5$.

(12) $r \geq 2$ and $p = 1$ or $q = 1$. $\tilde{R}_2^{[0]} = \mathfrak{sl}_r + \mathbb{R}^r + \mathbb{R}^{rp} + \mathbb{R}^p$, where $\mathbb{R}^p$ is in the center. $\tilde{R}_3^{[0]} = \mathfrak{sl}_r + \mathbb{R}^r + \mathbb{R}^{rp} + \mathfrak{sl}_p + \mathbb{R}^{pl_2}$, and finally

$$\tilde{R}_i^{[0]} = \mathfrak{sl}_r + \mathbb{R}^r + \mathbb{R}^{rp} + \mathfrak{sl}_p + \mathfrak{sl}_{l_2} + \mathbb{R}^{pl_2},$$

for all $i \leq 4$.

(13) $r \geq 2$ and $p, q > 1$. $\tilde{R}_2^{[0]} = \mathfrak{sl}_r + \mathbb{R}^{rq} + \mathbb{R}^{rp} + \mathbb{R}^{pq}$, where $\mathbb{R}^{pq}$ is in the center. $\tilde{R}_3^{[0]} = \mathfrak{sl}_r + \mathbb{R}^{rq} + \mathbb{R}^{rp} + \mathfrak{sl}_p + \mathfrak{sl}_q + \mathbb{R}^{pql_2}$, and finally

$$\tilde{R}_i^{[0]} = \mathfrak{sl}_r + \mathbb{R}^{rq} + \mathbb{R}^{rp} + \mathfrak{sl}_p + \mathfrak{sl}_q + \mathfrak{sl}_{l_2} + \mathbb{R}^{pql_2},$$

for all $i \leq 4$.

Since there are no other possibilities for the integers p, q and r, we have shown that the sequence $\tilde{R}_i^{[0]}$ stabilizes after at most 6 steps, which completes the proof. $\square$

REMARK 3.10.7. Actually several of these cases are impossible and the sequence $\tilde{R}_i^{[0]}$ stabilizes in less then 6 steps.

REMARK 3.10.8. We used that $l_i \geq 2$ only in cases 2, 5, 6, 7 and 11. If we had that $l_i = 1$ for all $i \geq 2$, we would have that the sequence $R_i^{[0]}$ does not stabilize, but it will at least weakly stabilize.

This together with $h(2) = 4$ suggests that it may be possible to construct relatively free Lie algebra on 2 generators in some nilpotent variety such which has not stabilizing tower of restricted derivations.

The next corollary is an immediate consequence of the theorem.

COROLLARY 3.10.9. The restricted derivation tower of $L[V, d]$ stabilizes after at most $\max\{3 + d/h(n), 6\}$ steps.

COROLLARY 3.10.10. If Conjecture 1.5.2 holds, then the restricted derivation tower of $L[V, d]$ stabilizes at the third level if $d \geq n + 1$ and $n|d - 1$.

REMARK 3.10.11. As in the case when the center of $\mathrm{Der}_0 L$ is trivial (see Remark 3.9.13), we have some description of the reductive part of the algebras DL^k The reductive part of DL^k is a semi simple Lie algebra which decomposes as a sum of copies of $\mathfrak{sl}_l$ for some l-es. It also acts faithfully on the nilpotent part and almost all nontrivial modules which appear in are either isomorphic to the standard one or its dual. Actually the same is the for the reductive part of $\tilde{R}_k^{[0]}$ but to show this one need to go over all cases in Lemma 3.10.6 and verify this statement.

CHAPTER 4

Automorphism Tower of Free Nilpotent Groups

In this chapter we study the automorphism tower of the free nilpotent group $\Gamma(n,d)$. The main tool in obtaining the description of this tower is to embed the groups $\mathrm{Aut}^k\,\Gamma(n,d)$ into a Lie group over (the reals) G^k and use methods form Lie algebra to study the automorphism group of G^k. After that we use a rigidity result proved in Grunewald and Platonov in [**7**], to show that we can embed the next group in automorphism tower into $\mathrm{Aut}\,G^k$.

Theorems 4.4.5 and 4.5.2 show that there is a close connection between the groups in the automorphism tower of $\Gamma(n,d)$, for $n \geq 3$, and the algebras in the tower of restricted derivations of the free Lie algebra $L[V,d]$. This allows us to derive the main result in this dissertation that the automorphism tower of the free nilpotent group stabilizes after finitely many steps (at least in the case of more than 2 generators).

The two generated case is slightly mote complicated, because we can not embed the groups $\mathrm{Aut}^k\,\Gamma$ for $k \geq 2$ into Lie groups. The reason for this is that the group $\mathrm{SL}_2(\mathbb{Z})$ has nontrivial cocycles. We go around this problem in section 4.6 by finding a characteristic subgroup $A\Gamma^k$ in $\mathrm{Aut}^k\,\Gamma$, which can be embedded in a Lie group G^k and first study the automorphism group of this subgroup.

4.1. Automorphism tower

For any group G we have a natural map from $\mathrm{ad} : G \to \operatorname{Aut} G$, which sends every element g to the inner derivation of the group G, given by conjugation with the element g.

DEFINITION 4.1.1. A group G is called compete if the map ad is an isomorphism, i.e., if every automorphism of G is inner and the center of G is trivial.

Iterating the construction of the automorphism group allows us to define the automorphism tower of the group G.

DEFINITION 4.1.2. Let G be a group, by the automorphism tower of G we mean the sequence

$$\operatorname{Aut}^0 G \to \operatorname{Aut}^1 G \to \operatorname{Aut}^2 G \to \cdots \to \operatorname{Aut}^k G \to \cdots ,$$

where $\operatorname{Aut}^0 G = G$ and $\operatorname{Aut}^{k+1} G = \operatorname{Aut}(\operatorname{Aut}^k G)$. The homomorphisms $\mathrm{ad}_k : \operatorname{Aut}^k G \to \operatorname{Aut}^{k+1} G$ comes form the adjoint action of $\operatorname{Aut}^k G$ on itself.

REMARK 4.1.3. If the group G has a trivial center then the map ad_0 is injection and it can easily be seen that all groups $\operatorname{Aut}^k G$ have trivial centers. Therefore $\operatorname{Aut}^k G$ form an increasing sequence of groups, which explains why Aut^k is called a tower.

DEFINITION 4.1.4. We say that the automorphism tower of the group G stabilizes at level k (or after k steps) if the map ad_k is an isomorphism. We also say that the tower weakly stabilizes at level k, if $\operatorname{Aut}^k G \simeq \operatorname{Aut}^{k+1} G$.

A classical result of Wielandt [**24**], asserts that the automorphism tower stabilizes for any finite group Γ with trivial center. Similar result for infinite groups with center does not hold in general.

Baumslag conjectured that for any finitely generated nilpotent group Γ, the automorphism tower of Γ stabilizes after finitely many steps. (Since this conjecture is not published, we will refer to the strongest version of the conjecture, i.e., that the tower stabilizes in the sense of Definition 4.1.4.)

In [**5**], Formanek and Dyer proved that the automorphism tower of the free nilpotent group of class 2 stabilizes at the second level, when the number of generators is different than 3. For the three generator case, the tower stabilizes at the third level. Similar results are known for other nilpotent groups, like the group of $n \times n$ upper triangular matrices, and in all these case the automorphism tower is very short.

In the rest of this chapter we study the automorphism tower of the free nilpotent groups $\Gamma(n, d)$.

4.2. Rigidity results

Our approach in studying the automorphism tower of the free nilpotent group $\Gamma(n, d)$ is to embed each group $\operatorname{Aut}^k \Gamma$ as a lattice into a Lie group over reals G^k.

We say that the pair (G, Γ) of Lie group and a lattice in it, is rigid if any automorphism of the lattice Γ comes from an automorphism of the Lie group G. (In section 4.6 we will see that not all groups in the automorphism tower of $\Gamma(n, d)$ can be embedded as a rigid lattices in a Lie group.)

The first rigidity results were proved my Margulis [**16**] and Mostow [**17**] for lattices in higher rank simple Lie groups. These results were latter generalized to some simple rank 1 groups.

All groups in the automorphism tower of the free nilpotent group $\Gamma(n, d)$ have big nilpotent part and they are very far from semi simple group, for which the classical rigidity results hold.

In a resent paper, F. Grunewald and Platonov [**7**] proved a rigidity result for a groups with large nilpotent radical. Here we will state one of the main results from their paper, which we will use in the next sections. This rigidity theorem is valid only for groups with strong unipotent radical.

Definition 4.2.1. Let G be a Lie group. We can write $G = U.H$, where U is the unipotent radical of G and H is a reductive group, which has trivial intersection with U. We say that G has a strong unipotent radical if the adjoint action of H on U is faithful.

For groups with strong unipotent radical we have the following rigidity result.

Theorem 4.2.2 (Grunewald, Platonov). *Let G be an algebraic group with strong unipotent radical and let Γ be a lattice in G. Then for any automorphism Φ of the lattice Γ there exists an automorphism ϕ of G and a map $v : \Gamma \to \mathrm{Cen}(U)$ such that*

$$\Phi(\gamma) = v(\gamma)\phi(\gamma),$$

for any $\gamma \in \Gamma$. Also we have that v is trivial on $\Gamma \cap U$.

Proof. Here we give a heuristic explanation why such result holds. For a complete proof we refer the readers to the original paper (see [**7**]).

Let denote $\mathrm{I}\Gamma = \Gamma \cap U$. This is a nilpotent group. We can reconstruct U starting from $\mathrm{I}\Gamma$ first by taking the Malcev completion of the nilpotent group $I\Gamma$, and then by completing the resulting arithmetic group over $\mathbb{Q}$ to a Lie group over $\mathbb{R}$. This allows us to lift any automorphism Φ of the group $\mathrm{I}\Gamma$ to an automorphism of $\tilde{\phi}$ of U.

If group G have strong unipotent radical then $G/\mathrm{Cen}(U)$ acts faithfully on U therefore we can embed $G/\mathrm{Cen}(U)$ into $\mathrm{Aut}\, U$. The image of this embedding coincides

with the Zariski closure of $\Gamma/\operatorname{Cen}(\mathrm{I}\Gamma)$ in $\operatorname{Aut} U$, where the last embedding comes from the adjoint action of Γ on $\mathrm{I}\Gamma$ and lifting the automorphisms for $\mathrm{I}\Gamma$ to automorphisms of U.

Let Φ is an automorphism of the group Γ, we can restrict Φ to an automorphism $\bar{\Phi}$ of $\mathrm{I}\Gamma$ and then lift $\bar{\Phi}$ to an automorphism $\bar{\phi}$ of U. This automorphism is an element of $\operatorname{Aut} U$ and it acts on $\operatorname{Aut} U$ by conjugation as an automorphism $\tilde{\phi}$.

By construction $\tilde{\phi}$ preserves the image of $\Gamma/\operatorname{Cen}(\mathrm{I}\Gamma)$ in $\operatorname{Aut} U$, therefore it preserves $G/\operatorname{Cen}(U)$. Thus we have obtained an automorphism $\tilde{\phi}$ of the group $G/\operatorname{Cen}(G)$ which we can restrict to the reductive part of the group G. This together with $\bar{\phi}$, allows us to construct an automorphism ϕ of the group G. By construction we have that for any $\gamma \in \mathrm{I}\Gamma$ we have that $\phi(\gamma) = \Phi(\gamma)$. Also this construction gives that the adjoint actions on $I\Gamma$ of $\phi(\gamma)$ and $\Phi(\gamma)$ on $I\Gamma$ coincide for any $\gamma \in \Gamma$. This shows that

$$\Phi(\gamma) = v(\gamma)\phi(\gamma),$$

for some map $v : \Gamma \to \operatorname{Cen} U$, which is trivial on $I\Gamma$. $\square$

REMARK 4.2.3. ϕ and v are not uniquely defined. If ϕ acts trivially on G/U then the map v is a cocycle and it is defined up to a cohomotopy. Therefore we say that if G has strong unipotent radical every automorphism of the lattice Γ comes from an automorphism of the Lie group up to a cocycle from the reductive part $\Gamma/(\Gamma \cap U)$ of Γ to the center of the group U. This gives that $\operatorname{Aut}\Gamma/(\operatorname{Aut}\Gamma \cap \operatorname{Aut} G)$ can be embedded into $H^1(\Gamma/(\Gamma \cap U, \operatorname{Cen}(U))$.

If the reductive part of G is a semi simple group, which is a product of higher rank simple Lie groups, then we have that all cohomology groups are trivial, which gives that any automorphism of Γ comes from an automorphism of the Lie group G.

Similar results hold if the semi simple part have $\mathrm{SL}_2(\mathbb{Z})$ as a quotient and $\mathrm{Cen}(U)$ decomposes as $\mathrm{SL}_2(\mathbb{Z})$ module as a sum of trivial modules and standard ones (see [**2**]).

4.3. Defining analogs of UL_k at group level

In the previous chapter we showed that the ideals UL_k play in important role in the description of the tower of restricted derivations of the free nilpotent Lie algebra. In this section we will define analogs of this ideals inside the group $\mathrm{Aut}\,\Gamma$.

Let us observe that the semi simple part of the group $\mathrm{Aut}\,\Gamma$ is $\mathrm{GL}_n(\mathbb{Z})$, therefore we need to construct the analogs of the ideals UL_k using the action of the group $\mathrm{GL}_n(\mathbb{Z})$, not the action of the Lie algebra $\mathfrak{sl}_n$.

Definition 4.3.1. Let $\widetilde{UL}^{+}$ (respectively $\widetilde{UL}^{[0]}$) denotes the maximal submodule of UL without $\mathrm{GL}_n(\mathbb{Z})$ invariant vectors (consisting entirely of invariant vectors). Let us construct the ideals $\widetilde{UL}_k$ of UL in the same way we constructed the ideals UL_k in definition 3.6.1, but starting form $\widetilde{UL}^{+}$ instead UL^{+}.

Now can define the group analogs of the ideals UL_k which will help us to construct the automorphism tower of the group $\Gamma(n, d)$.

Definition 4.3.2. Let $UG_k(n, d)$ be the Lie subgroup of $UG(n, d)$ obtained by exponentiating the ideal $\widetilde{UL}_k$.

Lemma 4.3.3. *For any k, $UG_k(n, d)$ is normal subgroup of $G^1(n, d)$, which is preserved by all automorphisms of this group.*

Proof. The Lie algebras corresponding to UG_k by construction are ideals in the Lie algebra $\mathrm{Der}_0\, L(n, d)$ corresponding to G^1. Therefore UG_k are normal subgroups.

These subgroups are invariant under all automorphisms of UG_k, because the ideals UL_k are characteristic ideals in UL and $\mathrm{Der}_0 L$ by Lemma 3.6.2. □

LEMMA 4.3.4. *UG_k form a decreasing sequence of normal subgroups of UG (and G^1), which stabilizes after at most $1 + d/h(n)$ steps.*

PROOF. The stabilization of this sequence of subgroups follows immediately from the stabilization their Lie algebras UL_k which is given by Theorem 3.6.4. □

LEMMA 4.3.5. *The factor group UG_0/UG_∞ is abelian.*

PROOF. The Lie algebra corresponding to this factor group is UL_0/UL_∞ which is abelian by construction. □

DEFINITION 4.3.6. Let us define the analogs of the spaces $U_{\mathrm{gen},k}$ using the Lie group UG. Let us denote $UG_{\mathrm{gen},k}(n,d) = UG_k(n,d)/UG_{k+1}(n,d)$ and the corresponding subgroups in the center of the UG:

$$CUG_k(n,d) = UG_k(n,d) \cap \mathrm{Cen}(UG(n,d)) \quad \text{and}$$

$$CUG_{\mathrm{gen},k}(n,d) = CUG_k(n,d)/CUG_{k+1}(n,d).$$

REMARK 4.3.7. All groups $UG_{\mathrm{gen},k}$, CUG_k and $CUG_{\mathrm{gen},k}$ are abelian and are naturally isomorphic to their Lie algebras.

Finally we need to define the discrete analogs of all these groups using the group $\mathrm{IAut}\,\Gamma(n,d)$.

DEFINITION 4.3.8. Let us define the following groups using $\mathrm{IAut}\,\Gamma(n,d)$:

- $\mathrm{IAut}\,\Gamma_k = \mathrm{IAut}\,\Gamma \cap UG_k$;

- $\mathrm{CIAut}\,\Gamma_k = \mathrm{IAut}\,\Gamma \cap CUG_k = \mathrm{IAut}\,\Gamma_k \cap \mathrm{Cen}(\mathrm{IAut}\,\Gamma)$;
- $\mathrm{IAut}\,\Gamma_{\mathrm{gen},k} = \mathrm{IAut}\,\Gamma_k / \mathrm{IAut}\,\Gamma_{k+1}$;
- $\mathrm{CIAut}\,\Gamma_{\mathrm{gen},k} = \mathrm{CIAut}\,\Gamma_k / \mathrm{CIAut}\,\Gamma_{k+1}$.

REMARK 4.3.9. The groups $\mathrm{IAut}\,\Gamma_k$ and $\mathrm{CIAut}\,\Gamma_k$ are normal subgroups of $\mathrm{Aut}\,\Gamma$ and $\mathrm{IAut}\,\Gamma$, which are invariant under all automorphisms. They also form two decreasing sequences of normal subgroups of length at most $1 + d/h(n)$.

LEMMA 4.3.10. *The groups* $C\,\mathrm{IAut}\,\Gamma_k$*,* $\mathrm{IAut}\,\Gamma_{\mathrm{gen},k}$ *and* $C\,\mathrm{IAut}\,\Gamma_{\mathrm{gen},k}$ *are free abelian and there is a natural action of* $\mathrm{GL}_n(\mathbb{Z})$ *on them.*

PROOF. The groups are free abelian groups because they are lattices in the corresponding Lie group which are abelian (without compact factors).

The group $\mathrm{Aut}\,\Gamma$ acts by conjugation on $\mathrm{IAut}\,\Gamma_k$. Its subgroup $\mathrm{IAut}\,\Gamma$ acts trivially modulo $\mathrm{IAut}\,\Gamma_{k+1}$ (because $[UL, UL_k] \subset UL_{k+1}$ by construction), therefore we can define action of the factor group $\mathrm{Aut}\,\Gamma / \mathrm{IAut}\,\Gamma$ on $\mathrm{IAut}\,\Gamma_k / \mathrm{IAut}\,\Gamma_{k+1}$. This defines the action of $\mathrm{GL}(\mathbb{Z})$ on $\mathrm{IAut}\,\Gamma_{\mathrm{gen},k}$, the action on the other groups are defined in similar way. □

4.4. Automorphism tower case with trivial center

In this section we will describe the automorphism tower of the free nilpotent group $\Gamma(n, d)$ in the case $n \geq 2$ and $\mathrm{Cen}\,\mathrm{Aut}\,\Gamma = 0$ (by a result of Formanek [**6**] this is equivalent to $2n \not| d - 1$). In this and in the next two section we will write Γ on instead of $\Gamma(n, d)$ to prevent to notations from becomming too complicated.

Let us start by first describing the second group in the automorphism tower.

The group $\mathrm{Aut}\,\Gamma$ is a lattice in the Lie group G^1. Since the semi simple part of $\mathrm{Aut}\,\Gamma$ is $\mathrm{GL}_n(\mathbb{Z})$ which do not have any non trivial cocycles if $n \geq 3$ (the case $n = 2$

is slightly more complicated — see the proof of Theorem 4.4.2) by the rigidity results we can embed $\mathrm{Aut}(\mathrm{Aut}\,\Gamma)$ into $\mathrm{Aut}\,G^1$. Therefore out first goal is to describe the automorphism group $\mathrm{Aut}\,G^1$.

THEOREM 4.4.1. *The group of outer automorphisms of* $G^1(n,d)$ *is isomorphic to*

$$(\mathbb{R}^*/\{\pm 1\}) \ltimes WG_0,$$

where

$$WG_0 \subset \mathrm{Hom}_{\mathrm{GL}(\mathbb{Z})}(UG_{\mathrm{gen},0}, CUG_0)/\big(UG^{(d-2)}/\,\mathrm{Cen}(UG)\big)^{\mathrm{GL}(\mathbb{Z})}$$

is the subset consisting of all map whose projection in $\mathrm{End}(CUG_{\mathrm{gen},0})$ *is invertible, here the projection from* $\mathrm{Hom}(UG_{\mathrm{gen},0}, CUG_0)$ *to* $\mathrm{End}(CUG_{\mathrm{gen},0})$ *is defined using the natural embedding of* $CUG_{\mathrm{gen},0}$ *in* $UG_{\mathrm{gen},0}$ *and the natural projection from* CUG_0 *to* $CUG_{\mathrm{gen},0}$*. The group structure on the set* WG_0 *comes from the composition of the maps through* $CUG_{\mathrm{gen},0}$*.*

The group $\big(UG^{(d-2)}/\,\mathrm{Cen}(UG)\big)^{\mathrm{GL}(\mathbb{Z})}$ *is the group of all inner derivations which lie in*

$\mathrm{Hom}(UG_{\mathrm{gen},0}, CUG_0)$*. Finally* $\mathbb{R}^*/\{\pm 1\}$ *comes from* $\mathrm{GL}(\mathbb{R})/\,\mathrm{SL}^{\pm 1}(\mathbb{R})$

PROOF. The Lie algebra of the group $\mathrm{Aut}\,G^1$ is a subalgebra of the derivation algebra of G^1. Our first goal is to describe this subalgebra.

By Theorem 3.8.11 the space of outer derivations of the Lie algebra of the group G^1 is isomorphic to

$$Out = \mathbb{R} + \mathrm{Hom}_{\mathfrak{sl}}(UL_{\mathrm{gen},0}, CUL_0)/(UL^{(d-2)})^{\mathfrak{sl}},$$

because if $n \geq 3$ the spaces T is trivial. But the group G^1 has two connected components, therefore we need to take the subalgebra which preserved by the action of the outer automorphism, which comes from the conjugation with an element in the

connected component of G^1 which does not contain the identity. The subalgebra of *Out* preserved by this automorphism consists of all maps which preserve the action of $\mathrm{GL}(\mathbb{Z})$.

Now we only need to exponentiate this Lie algebra in order to construct the Lie group. Exponentiating $\mathbb{R}$ gives us the extension of the semi simple part form $\mathrm{SL}_n^{\pm 1}(\mathbb{R})$ to $\mathrm{GL}_n(\mathbb{R})$, which corresponds to the factor $\mathbb{R}^*/\{\pm 1\}$ in the outer automorphism group. Exponentiation of the other part

$$\mathrm{Hom}_{\mathrm{GL}(\mathbb{Z})}(\widetilde{UL}_{\mathrm{gen},0}, \widetilde{CUL}_0)/(UL^{(d-2)})^{\mathrm{GL}(\mathbb{Z})}$$

gives the group WG_0. $\square$

We can use the description of $\mathrm{Aut}\, G^1$, to obtain a description of $\mathrm{Aut}(\mathrm{Aut}\,\Gamma))$.

THEOREM 4.4.2. *The group of outer automorphisms of* $\mathrm{Aut}\,\Gamma$ *can be embedded into*

$$\mathrm{Hom}_{\mathrm{GL}(\mathbb{Z})}(\mathrm{IAut}\,\Gamma_{\mathrm{gen},0}, \mathrm{CIAut}\,\Gamma_0)/\big(\,\mathrm{IAut}\,\Gamma^{(d-2)}/\,\mathrm{Cen}(\mathrm{IAut}\,\Gamma)\big)^{\mathrm{GL}(\mathbb{Z})}$$

The image of this embedding consists of all maps, whose projection ϕ *in the space* $\mathrm{End}_{\mathrm{GL}(\mathbb{Z})}(\mathrm{CIAut}_{\mathrm{gen},0})$ *is invertible and it is possible to lift* ϕ *to an automorphism* $\tilde{\phi}$ *of* G^1 *which preserves the group* $\mathrm{Aut}\,\Gamma$*.*

PROOF. First we want to prove that $\mathrm{Aut}(\mathrm{Aut}\,\Gamma)$ can be embedded into the Lie group $\mathrm{Aut}\, G_1$. In order to prove that we need to verify that the pair $(G^1, \mathrm{Aut}\,\Gamma)$ satisfies the condition of Theorem 4.2.2.

The group G_1 has a strong unipotent radical, because $SL_n^{\pm 1}(\mathbb{R})$ acts faithfully on UG (since the action of $\mathfrak{sl}$ on UL is not trivial). Therefore we can lift any automorphism of $\mathrm{Aut}\,\Gamma$ to an automorphism of G^1 up to a cocycle γ from $\mathrm{GL}_n(\mathbb{Z})$ to $\mathrm{Cen}(UG)$.

If we have $n \geq 3$ then every cocycle of $\mathrm{GL}_n(\mathbb{Z})$ is homologically trivial. In the case $n = 2$, we have that $\mathrm{Cen}(UL) = \sum_k V^{[2k+1]}$ as $\mathrm{SL}_2(\mathbb{Z})$ module because it lies in the homogeneous component of odd degree. But the group $\mathrm{GL}_2(\mathbb{Z})$ does not have homologically non trivial cocycles in the module U unless U contains submodules corresponding to a partition $\lambda = [2k]$ for $k \geq 2$ (see [**2**]). This implies that γ is homologically trivial, therefore we may assume that γ is trivial and that $\mathrm{Aut}(\mathrm{Aut}\,\Gamma)$ is a subgroup of $\mathrm{Aut}(G^1(n,d))$. In order to describe the image it is enough to find all outer automorphisms of G^1 which preserve $\mathrm{Aut}\,\Gamma$.

Let $\phi \in WG_0$ be an outer automorphism, in order to preserve $\mathrm{Aut}\,\Gamma$ its image has to be in $\mathrm{Cen}(\mathrm{IAut}\,\Gamma)$ therefore ϕ is an element in

$$\mathrm{Hom}_{\mathrm{GL}(\mathbb{Z})}(\mathrm{IAut}\,\Gamma_{\mathrm{gen},0}, \mathrm{CIAut}\,\Gamma_0)/\left(\mathrm{IAut}\,\Gamma^{(d-2)}/\,\mathrm{Cen}(\mathrm{IAut}\,\Gamma)\right)^{\mathrm{GL}(\mathbb{Z})}.$$

For an element ϕ to preserve the lattice $\mathrm{Aut}\,\Gamma$ it is necessary that its projection in $\mathrm{End}_{\mathrm{GL}(\mathbb{Z})}(\mathrm{CIAut}_{\mathrm{gen},0})$ is invertible. A sufficient condition for ϕ to preserve this lattice is that ϕ lies in some unipotent subgroup of WG_0. Therefore using that $\mathrm{SL}(\mathbb{Z})$ is generated by the elementary matrices we can see that if for any partition λ we have $\det \phi\, \mathrm{End}_{\mathrm{GL}(\mathbb{Z})}(\mathrm{CIAut}^{\lambda}_{\mathrm{gen},0}) = 1$. Then ϕ can be lifted to an isomorphism of $\mathrm{Aut}\,\Gamma$. $\square$

Let G^2 be the Lie group corresponding to $\mathrm{Aut}^2\,\Gamma$, i.e., we define G^2 as the Zariski closure of $\mathrm{Aut}^2\,\Gamma$ in the Lie group $\mathrm{Aut}\,G^1$. The next lemma describes the Lie algebra of the group G^2. From this result we can notice that the Lie algebra of G^2 is almost the same as the second algebra in the restricted derivation tower of $L[V,d]$ which justifies the study of this tower.

REMARK 4.4.3. The Lie algebra of the group G^2 is

$$\mathfrak{sl} + UL + \mathrm{Hom}^0_{\mathrm{GL}(\mathbb{Z})}(\widetilde{UL}_{\mathrm{gen},0}, \widetilde{CUL}_0).$$

PROOF. The Lie algebra contains the subalgebra of W_0 generated by all nilpotent derivations, therefore it contains $R_0^\lambda \cap W_0$. This algebra is also a sub algebra of $\mathrm{Der}_0 \mathrm{Lie} G_1$, which implies that the Lie algebra of G^2 is exactly

$$\mathfrak{sl} + UL + \mathrm{Hom}^0_{\mathrm{GL}(\mathbb{Z})}(\widetilde{UL}_{\mathrm{gen},0}, \widetilde{CUL}_0).$$

By Remark 3.8.14 the semi simple part of this Lie algebra is a sum of copies of $\mathfrak{sl}_l$ for some l-es and it acts faithfully on the nilpotent part. Also almost all nontrivial modules which appears in the nilpotent part are either isomorphic to the standard modules or their duals. □

REMARK 4.4.4. We do not have a good description of the connected components of G^2, in order to obtain such description we need a detailed knowledge of how the group $\mathrm{Aut}\,\Gamma$ sits inside $\mathrm{Aut}\,G$. For example it can be shown that G^2 has two connected components if $d \leq n(n-1)$ unless $n = 3, d = 2$, when it has 4 connected components (for $d = 2$ this follows from a result of Formanek and Dyer [**4**], about the description of the automorphism tower of free nilpotent groups of class 2).

In section 3.9 we used the description of the second derivation algebra of $L[V, d]$ as a base of induction to describe all algebras in the tower of restricted derivation of $L[V, d]$. We can do the same for the case of discrete group Γ an its automorphism tower.

We can use the description of the second automorphism group of $\Gamma(n, d)$ as a base for induction which give us the structure of every group in the automorphism tower of the group Γ.

THEOREM 4.4.5. *a) The group of outer automorphisms of G^k is isomorphic to*

$$\left(\mathbb{R}^*/\{\pm 1\}\right) \ltimes WG_{k-1},$$

where

$$WG_{k-1} \subset \mathrm{Hom}_{\mathrm{GL}(\mathbb{Z})}(UG_{\mathrm{gen},k-1}, CUG_{k-1})$$

is the subset consisting of all map whose projection in $\mathrm{End}(CUG_{\mathrm{gen},k-1})$ *is invertible, where the multiplication on this space is given by a composition trough* $CUG_{\mathrm{gen},k-1}$.

b) The group of outer automorphisms of $\mathrm{Aut}^k \Gamma$ *can be embedded in*

$$\mathrm{Hom}_{\mathrm{GL}(\mathbb{Z})}(\mathrm{IAut}\,\Gamma_{\mathrm{gen},k-1}, \mathrm{CIAut}\,\Gamma_{k-1}).$$

The image of this embedding consists of all maps ϕ*, whose projection of in the space* $\mathrm{End}_{\mathrm{GL}(\mathbb{Z})}(C\,\mathrm{IAut}_{\mathrm{gen},k-1})$ *is invertible and the lifting of* ϕ *to an automorphism* $\tilde{\phi}$ *of* $\mathrm{SL}^{\pm 1}(\mathbb{R}) \ltimes UG_0$ *preserves the projection of* $\mathrm{Aut}\,\Gamma$ *into this Lie group.*

c) The automorphism group of $\mathrm{Aut}^k \Gamma$ *can be embedded into* $\mathrm{Aut}\, G^k$*. The Lie algebra of the Zariski closure* G^{k+1} *of* $\mathrm{Aut}^k \Gamma$ *is isomorphic to*

$$\mathfrak{sl} + UL + \sum_{i=0}^{k-1} \mathrm{Hom}^0_{\mathrm{GL}(\mathbb{Z})}(\widetilde{UL}_{\mathrm{gen},i}, \widetilde{CUL}_i).$$

Proof. a) The proof is by induction. Let us first describe the automorphism group of G^k. Using the lemma from the proof of Theorem 3.9.1 we can see that the algebra of outer derivations which are preserved by the action of the outer automorphism of order 2 obtained by conjugation by elements in the connected component of G^k corresponding to $\mathrm{SL}_n^{-1}(\mathbb{R})$ is

$$\mathbb{R} + \mathrm{Hom}_{\mathrm{GL}(\mathbb{Z})}(\widetilde{UL}_{\mathrm{gen},k-1}, \widetilde{CUL}_{k-1}).$$

From Lemma 3.9.3 it follows that this algebra is preserved by all other outer automorphisms of the Lie algebra of G_k coming from the other connected components. Therefore it coincides with the Lie algebra of the group of outer automorphisms of G_k.

We can obtain the group WG_k by exponentiating this Lie algebra as we did in Theorem 4.4.1 for the group $\operatorname{Aut} G^1$.

b) In order to prove that $\operatorname{Aut}^{k+1}\Gamma$ can be embedded in $\operatorname{Aut} G_k$ we need to verify that the pair $(G^k, \operatorname{Aut}^k \Gamma)$ satisfies the condition in Theorem 4.2.2. The group G^k has strong unipotent radical, because the semi simple part of its Lie algebra acts faithfully on the nilpotent part by Remark 3.9.13.

Now we need to verify that there are no homologically nontrivial cocycles from the semi simple part of the group $\operatorname{Aut}^k\Gamma$ to the center V of the unipotent part of the group G^k. All simple factors S of $\operatorname{Aut}^k\Gamma$ are isomorphic to $\mathrm{SL}_l(\mathbb{Z})$ or $\mathrm{GL}_l(\mathbb{Z})$ for different l-es. Unless S is original copy of $\mathrm{SL}_n(\mathbb{Z})$ then under the action of S, the space V decomposes as a sum of modules which are either trivial or isomorphic to the standard module or its dual, therefore every cocycle from S to V is homologically trivial. If S is the original copy of $\mathrm{GL}_n(\mathbb{Z})$ then as we saw in the proof of Theorem 4.4.2 there are no nontrivial cocycles.

This shows that $\operatorname{Aut}(\operatorname{Aut}\Gamma) \subset \operatorname{Aut} G^k$, and in order to describe the group of outer automorphisms of $\operatorname{Aut}^k\Gamma$ we only need to see which elements in WG_{k-1} preserve the lattice $\operatorname{Aut}^k\Gamma$. The argument is the same as the argument in the case for the second automorphism group.

c) The Lie algebra of G^{k+1} contains the subalgebra of W_{k-1} generated by all nilpotent derivations, therefore it contains $R_k^\lambda \cap W_k$. This algebra is also a sub algebra of $\operatorname{Der}_0 \operatorname{Lie} G^k$, which implies that the Lie algebra of G^{k+1} is exactly

$$\mathfrak{sl} + UL + \sum_{i=0}^{k-1} \operatorname{Hom}^0_{\mathrm{GL}(\mathbb{Z})}(\widetilde{UL}_{\mathrm{gen},i}, \widetilde{CUL}_i).$$

Notice that by Remark 3.9.13 the semi simple part of this Lie algebra acts faithfully on the nilpotent part. $\square$

An immediate corollary of this theorem is the following result about the stabilization of the automorphism tower of the free nilpotent groups.

THEOREM 4.4.6. *If* $2n \not| d-1$ *then the automorphism tower of the free nilpotent group* $\Gamma(n,d)$ *stabilizes after at most* $2+d/h(n)$ *steps, in particular it has finite height.*

REMARK 4.4.7. The height of the automorphism tower of Γ is usually the same as the length of the sequence $\widetilde{UL}_k$ (more precisely the maximal k such that the space $\mathrm{Hom}_{\mathrm{GL}(\mathbb{Z})}(\widetilde{UL}_{\mathrm{gen},k}, \widetilde{CUL}_k)$ is not trivial). The height of the automorphism tower may be smaller then the length of this sequence if and only if for the maximal k such that $\widetilde{UL}_{\mathrm{gen},k}$ is not trivial we have that

$$\widetilde{UL}^{\lambda}_{\mathrm{gen},k} = \widetilde{CUL}^{\lambda}_{\mathrm{gen},k} = \widetilde{CUL}^{\lambda}_{k}$$

and this is a simple $\mathfrak{sl}$ module corresponding to the partition λ, for all λ such that

$$\mathrm{Hom}_{\mathrm{GL}(\mathbb{Z})}(\widetilde{UL}^{\lambda}_{\mathrm{gen},k}, \widetilde{CUL}^{\lambda}_{k}) \neq 0.$$

In this case it is possible that the height of automorphism tower is the length of the sequence UL_k minus 1.

In particular if $d \leq 2n$ the automorphism tower stabilizes at the second or the third level. If the tower does not stabile at the second level then the group of outer automorphisms $\mathrm{Aut}^2\Gamma/\mathrm{Aut}^1\Gamma$ is cyclic group of order 2. It can be shown that unless $d=2$ and $n=2$ the tower stabilizes at the second level. Similarly if $2n < d \leq h(n)$ and $2n \not| d-1$, then the automorphism tower stabilizes at the third level.

REMARK 4.4.8. This result is generalization of the result by Formanek and Dyer for $d=2$, however in the case $d=2$ their result is more precise because it says that the tower stabilizes after 2 steps unless $n=2$ and our result gives that it stabilizes after at most 3 steps and that the quotient $\mathrm{Aut}^2\Gamma/\mathrm{Aut}\,\Gamma$ is finite.

4.5. Automorphism tower case with center

In the previous section we obtained a description of the automorphism tower of the group Γ in the case when $\mathrm{Cen}(\mathrm{Aut}\,\Gamma)$ is trivial. Now we want to consider the case when the center of the first group in the tower is not trivial.

There is a close analogy between this case and the tower of restricted derivations of $L[V,d]$ if the first algebra has a center, for example all groups in the automorphism tower of Γ splits as a direct product of two groups.

In this section we will assume that $n \geq 3$ which allows us to embed every group in the automorphism tower in some Lie group. The case of 2 generated groups will be considered in the next section.

Let us start with the second group in the tower. As in the case without center we first describe the automorphism group of G^1 and show that $\mathrm{Aut}^2\,\Gamma$ can be embedded into this group.

THEOREM 4.5.1. *a) The automorphism group of G^1 splits as*

$$\mathrm{Aut}\,G^1 = (\mathbb{R}^*/\{\pm 1\}) \ltimes (\widetilde{NG}^2 \times \widetilde{CG}^2),$$

where the group $\widetilde{NG}^2$ is the extension of the image of the group of inner automorphisms by

$$WG_0 \subset \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Z})}(UG_{\mathrm{gen},0}, CUG_0),$$

where WG_0 is the subset consisting of all maps whose projection in $\mathrm{End}(CUG_{\mathrm{gen},0})$ is invertible and

$$CG^2 \subset \mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Z})}(UG^{[0]}_{\mathrm{gen}}, \mathrm{Cen}(G^1)).$$

b) The automorphism group of $\mathrm{Aut}\,\Gamma$ splits as a direct product $\mathrm{Aut}^2\,\Gamma = N\Gamma^2 \times C\Gamma^2$, where $N\Gamma^2$ and $C\Gamma^2$ are discrete subgroups of $\widetilde{NG}^2$ and $\widetilde{CG}^2$. The group $N\Gamma^2$

is an extension of $\operatorname{Aut}\Gamma/\operatorname{Cen}(\operatorname{Aut}\Gamma)$ *by the subgroup of*

$$\operatorname{Hom}_{\mathrm{GL}(\mathbb{Z})}(\mathrm{IAut}\,\Gamma_{\mathrm{gen},0}, \mathrm{CIAut}\,\Gamma_0)$$

consisting of all maps ϕ*, whose projection in* $\operatorname{End}_{\mathrm{GL}(\mathbb{Z})}(\mathrm{CIAut}_{\mathrm{gen},0})$ *can be lifted to automorphism of* $\tilde{\phi}$ *which preserves* $\operatorname{Aut}\Gamma$*. The group* $C\Gamma^2$ *is*

$$C\Gamma^2 \subset \operatorname{Hom}_{\mathrm{GL}_n(\mathbb{Z})}(\mathrm{IAut}\,\Gamma_{\mathrm{gen}}^{[0]}, \operatorname{Cen}(\operatorname{Aut}\Gamma))$$

c) The Zariski closure G^2 *of the group* $\operatorname{Aut}^2\Gamma$ *in* $\operatorname{Aut}G^1$ *splits as a direct product* $G^2 = NG^2 \times CG^2$*, and the Lie algebra of* G^2 *is isomorphic to*

$$\mathfrak{sl}+UL/\operatorname{Cen}(UL)^{\mathrm{GL}(\mathbb{Z})}+\operatorname{Hom}^0_{\mathrm{GL}(\mathbb{Z})}(\widetilde{UL}_{\mathrm{gen},0}, \widetilde{CUL}_0)\oplus\operatorname{Hom}^0_{\mathrm{GL}(\mathbb{Z})}(\widetilde{UL}^{[0]}_{\mathrm{gen}}, \operatorname{Cen}(UL)^{\mathrm{GL}(\mathbb{Z})}).$$

Proof. a) The situation is similar to Theorems 4.4.1. The automorphism group of G^1 splits because the its Lie algebra is a direct sum of two components (see Theorem 3.8.16).

b) The automorphism group of $\operatorname{Aut}\Gamma$ can be embedded into $\operatorname{Aut}G^1$, because the group G^1 has strong unipotent radical and GL_n goes not have any homologically non trivial cocycles (see Theorem 4.4.2). The group splits as a direct product because automorphisms which acts trivially on $\operatorname{Cen}(\operatorname{Aut}\Gamma)$ commutes with automorphism which acts trivially on $\operatorname{Aut}\Gamma/\operatorname{Cen}(\operatorname{Aut}\Gamma)$.

c) The group G^2 splits as a direct product because the group $\operatorname{Aut}^2\Gamma$ does. We can obtain the description of its Lie algebra from Theorems 4.4.1 and 3.8.16.

□

The group $\operatorname{Aut}^2\Gamma$ is a very good model for all group in the automorphism tower. We use the previous theorem as a base for induction which will describe all groups in the automorphism tower of $\Gamma(n,d)$.

THEOREM 4.5.2. *a) The automorphism group of G^k splits as a direct product*

$$\operatorname{Aut} G^k = \big(\mathbb{R}^*/\{\pm 1\}\big) \ltimes (\widetilde{NG}^{k+1} \times \widetilde{CG}^{k+1}),$$

where the group $\mathbb{R}^*/\{\pm 1\} \ltimes \widetilde{NG}^{k+1}$ is the automorphism group of NG^k, also $\widetilde{NG}^{k+1}$ is an extension of NG^k by

$$WG_{k-1} = \operatorname{Hom}_{\mathrm{GL}_n(\mathbb{Z})}(UG_{\mathrm{gen},k-1}, CUG_{k-1}),$$

where WG_{k-1} is the subset consisting of all maps whose projection in $\operatorname{End}(CUG_{\mathrm{gen},k-1})$ is invertible and

$$\widetilde{CG}^{k+1} = \operatorname{Aut} CG^k \ltimes \operatorname{Hom}\big(NG^k/[NG^k, NG^k], \operatorname{Cen}(CG^k)\big).$$

b) The automorphism group of $\operatorname{Aut}^k \Gamma$ splits as a direct product $\operatorname{Aut}^{k+1}\Gamma = N\Gamma^{k+1} \times C\Gamma^{k+1}$, where $N\Gamma^{k+1}$ and $C\Gamma^{k+1}$ are discrete subgroups of $\widetilde{NG}^{k+1}$ and $\widetilde{CG}^{k+1}$. The group $N\Gamma^{k+1}$ is an extension of $N\Gamma^k$ by the subgroup of

$$\operatorname{Hom}_{\mathrm{GL}(\mathbb{Z})}(\mathrm{IAut}\,\Gamma_{\mathrm{gen},k-1}, \mathrm{CIAut}\,\Gamma_{k-1})$$

consisting of all maps ϕ, whose projection in $\operatorname{End}_{\mathrm{GL}(\mathbb{Z})}(\mathrm{CIAut}_{\mathrm{gen},0})$ can be lifted to automorphism of $\tilde{\phi}$ which preserves $\operatorname{Aut}\Gamma$. The group $C\Gamma^{k+1}$ is isomorphic to

$$\operatorname{Aut} C\Gamma^k \ltimes \operatorname{Hom}(N\Gamma^k/[N\Gamma^k, N\Gamma^k], \operatorname{Cen}(C\Gamma^k))$$

c) The Zariski closure G^k of the group $\operatorname{Aut}^{k+1}\Gamma$ in $\operatorname{Aut} G^k$ splits as a direct product $G^{k+1} = NG^{k+1} \times CG^{k+1}$, and the Lie algebra of NG^{k+1} is isomorphic to

$$\mathfrak{sl} + UL/\operatorname{Cen}(UL)^{\mathrm{GL}(\mathbb{Z})} + \sum_{i=0}^{k-1} \operatorname{Hom}^0_{\mathrm{GL}(\mathbb{Z})}(\widetilde{UL}_{\mathrm{gen},i}, \widetilde{CUL}_i).$$

PROOF. The proof of this theorem is by induction on k, where Theorem 4.5.1 serves as a base case. The induction step is the same as in Theorem 4.4.5 the case when the center of $\operatorname{Aut}\Gamma$ is trivial. The only thing that have to be added is the

splitting of the group $Auk^k\Gamma$ which follows from the splitting of the algebras in the tower of restricted derivations. □

As in the case when the center is trivial we have that the sequence $N\Gamma^k$ stabilizes after finitely many steps, because $\mathrm{IAut}\,\Gamma_{\mathrm{gen},k}$ are trivial for $k \geq 1 + d/h(n)$. However this is not enough to claim that the automorphism tower of $\Gamma(n,d)$ stabilizes.

The stabilization of the split part $C\Gamma^k$ does not follow the stabilization of $\tilde{R}_k^{[0]}$, since we do not have that $\tilde{R}_k^{[0]}$ is the Lie algebra of CG^k. Before starting the proof of the stabilization of $C\Gamma^k$ we need a technical Lemma similar to Lemma 3.10.5

LEMMA 4.5.3. *The abelinization of the group $N\Gamma^k$ is*

$$N\Gamma^k/[N\Gamma^k, N\Gamma^k] = \mathbb{Z}^{l_k} \times \mathbb{F}_2^{m_k},$$

where $l_k \geq 2$ and $m_k \geq 1$, unless $C\Gamma^2$ is trivial. The cyclic factors comes from the copies of $\mathrm{GL}_l(\mathbb{Z})$ which appear in the semi simple part of the group $\mathrm{Aut}^k\,\Gamma$.

PROOF. Form the construction of the group NG^k it follows that their abelinization is a sum of several copies of $\mathbb{Z}$ and cyclic groups of order 2, therefore we have that the above isomorphism for some numbers l_k and m_k. Lemma 3.10.5 gives that $l_k \geq 2$. $m_k \geq 1$ because $N\Gamma^k$ has $\mathrm{GL}_n(\mathbb{Z})$ as a quotient therefore its abelinization contains at least one element of order 2. □

Now we are ready to proof the stabilization of the sequence $C\Gamma^k$.

LEMMA 4.5.4. *The sequence*

$$C\Gamma^2 \to C\Gamma^3 \to C\Gamma^4 \to cdots$$

stabilizes after at most 5 steps.

Proof. The behavior of this sequence is determined by the dimensions of the ranks of the free abelian groups $\mathrm{IAut}\,\Gamma^{[0]}_{\mathrm{gen}}$, $\mathrm{Cen}(\mathrm{Aut}\,\Gamma)$ and $\mathrm{Cen}(\mathrm{Aut}\,\Gamma) \cap [\mathrm{IAut}\,\Gamma, \mathrm{IAut}\,\Gamma]$. Let us define p, q, r as follows

$$p = \mathrm{rank}(\mathrm{IAut}\,\Gamma/(\mathrm{Cen}(\mathrm{IAut}\,\Gamma)[\mathrm{IAut}\,\Gamma, \mathrm{Aut}\,\Gamma]), \quad q = \mathrm{rank}\,\mathrm{Cen}(\mathrm{Aut}\,\Gamma \cap [\mathrm{IAut}\,\Gamma, \mathrm{IAut}\,\Gamma])$$

and

$$r = \mathrm{rank}(\mathrm{Cen}(\mathrm{Aut}\,\Gamma)/\,\mathrm{Cen}(\mathrm{Aut}\,\Gamma) \cap [\mathrm{IAut}\,\Gamma, \mathrm{IAut}\,\Gamma]).$$

We will use l_i (and m_i) to denote the rank (and the dimension of the torsion part of the group $N\Gamma^k/[N\Gamma^k, N\Gamma^k]$. Denote $s_i = l_i + p_i$.

Let us consider the different possibilities for this dimensions:

(1) $p = r = 0$ or $r = q = 0$. In this case all groups $C\Gamma^i$, for $i \geq 2$ are trivial and the sequence stabilizes.

(2) $r = 0$, $p = q = 1$. In this case we case we have that $C\Gamma^2 = \mathbb{Z}$ is abelian; this implies that $C\Gamma^3 = \mathbb{Z}^{l_2}$ is also abelian and $C\Gamma^4 = \mathrm{GL}_{l_2}(\mathbb{Z}) \ltimes \mathbb{Z}^{l_2 l_3}$ and finally $C\Gamma^i = \mathrm{GL}_{l_2}(\mathbb{Z}) \ltimes \mathbb{Z}^{l_2 l_3} \rtimes \mathrm{GL}_{l_3}(\mathbb{Z})$ for $i \geq 5$.

(3) $r = 0$ and $p.q > 1$ In this case we case we have $C\Gamma^2 = \mathbb{Z}^{pq}$ is abelian, therefore $C\Gamma^3 = \mathrm{GL}_{pq}(\mathbb{Z}) \ltimes \mathbb{Z}^{pql_2}$ and finally $C\Gamma^i = \mathrm{GL}_{pq}(\mathbb{Z}) \ltimes \mathbb{Z}^{pql_2} \rtimes \mathrm{GL}_{l_2}(\mathbb{Z})$ for $i \geq 4$.

(4) $r = 1$ and $p = q = 0$. In this case we have two possibilities. The first one is $C\Gamma^2 = \{1\}$, in which case all groups $C\Gamma^i$ are trivial. The other is $C\Gamma^2 = \mathbb{F}_2$, which leads to $C\Gamma^3 = \mathbb{F}^{s_2}$ and $C\Gamma^4 = \mathrm{SL}_{s_2}(\mathbb{F}_2) \ltimes \mathbb{F}_2^{s_2 s_3}$ and $C\Gamma^i = \mathrm{SL}_{s_2}(\mathbb{F}_2) \ltimes \mathbb{F}_2^{s_2 s_3} \rtimes \mathrm{SL}_{s_2}(\mathbb{F}_2)$ for all $i \geq 4$.

(5) $r = 1$ and $p = 0$ or $q = 0$ (say $q = 0$). This case again splits into two subcases — in the first one we have $C\Gamma^2 = \mathbb{Z}^p$, therefore the sequence $C\Gamma^i$ stabilizes as in cases 2 and 3. The other possibility is $C\Gamma^2 = \mathbb{F}_2 \ltimes \mathbb{Z}^p$, which has trivial center. If $p = 1$ this is the infinite dihydric group and it has

non trivial automorphisms, i.e., $C\Gamma^i = C\Gamma^2$ for all i. If $p \geq 1$ we have that $C\Gamma^i = \mathrm{GL}_p(\mathbb{Z}) \ltimes Z^p$ for all $i \geq 3$.

(6) $r = 1$ and $p = q = 1$. In this case $C\Gamma^2$ is either the free two generated nilpotent group of class 2, or its extension by $\mathbb{F}_2$ in either case we have that $C\Gamma^3 = \mathrm{GL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^2 \times Z^{l_2})$, where GL acts naturally on $\mathbb{Z}^2$ and acts by multiplication with the determinant on $\mathbb{Z}^{l_2}$, therefore we have that $C\Gamma^i = (\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_{l_2}(\mathbb{Z})) \ltimes (\mathbb{Z}^2 \times Z^{l_2})$, for all $i > 3$.

(7) $r = 1$ and $p > 1$ or $q > 1$. In this case $C\Gamma^2$ is either the nilpotent group of class 2 having $\mathbb{Z}^{p+q}$ as abelinization and $\mathbb{Z}^{pq}$ as center, or its extension by $\mathbb{F}_2$. In either case we have that $C\Gamma^3 = (\mathrm{GL}_p(\mathbb{Z}) \times \mathrm{GL}_q(\mathbb{Z})) \ltimes (\mathbb{Z}^p \times \mathbb{Z}^g \times Z^{pql_2})$, since this group has trivial center we have that $C\Gamma^i = (\mathrm{GL}_p(\mathbb{Z}) \times \mathrm{GL}_q(\mathbb{Z})) \times \mathrm{GL}_{l_2}(\mathbb{Z})) \ltimes (\mathbb{Z}^q \times \mathbb{Z}^p \times Z^{pql_2})$, for all $i \geq 3$.

(8) $r \geq 2$ and $p = q = 0$. $C\Gamma^2 = \mathrm{SL}_r(\mathbb{Z})$ or $C\Gamma^2 = \mathrm{SL}_r(\mathbb{Z})$. Depending on the parity of r we have that $\mathrm{Cen}(C\Gamma^2)$ is either trivial or cyclic of order 2, in the first case we have $C\Gamma^i$ is PSL_r or PGL_r for all $r \geq 3$. In the second we have $C\Gamma^i = PSL_r(\mathbb{Z}) \times \mathbb{F}_2^{s_2}$, and as in case 4 the sequence $C\Gamma^i$ stabilizes after 5 steps.

(9) $r \geq 2$ and $p = 0$ or $q = 0$ (say $q = 0$). In this case $C\Gamma^2 = \mathrm{SL}_r(\mathbb{Z}) \ltimes \mathbb{Z}^{rp}$ or $C\Gamma^2 = \mathrm{GL}_r(\mathbb{Z}) \ltimes \mathbb{Z}^{rp}$. In both cases we have that $C\Gamma^3 = (\mathrm{GL}_r(\mathbb{Z}) \times \mathrm{GL}_p(\mathbb{Z})) \ltimes \mathbb{Z}^{rp}$ and all others $C\Gamma^i$ are equal to $C\Gamma^3$.

(10) $r \geq 2$ and $p, q \geq 1$. In this case $C\Gamma^2$ is an extension of a nilpotent group of class 2 having $\mathbb{Z}^{r(p+q)}$ as abelinization and $\mathbb{Z}^{pq}$ as center, by either $\mathrm{SL}_r(\mathbb{Z})$ or $\mathrm{GL}_z(\mathbb{Z})$. In both cases we have that $C\Gamma^3 = (\mathrm{GL}_r(\mathbb{Z}) \times \mathrm{GL}_p(\mathbb{Z}) \times \mathrm{GL}_q(\mathbb{Z})) \ltimes$

$(\mathbb{Z}^{rp} \times \mathbb{Z}^{rq} \times \mathbb{Z}^{pql_2})$ and all other $C\Gamma^i$ are equal to

$$C\Gamma^3 = (\mathrm{GL}_r(\mathbb{Z}) \times \mathrm{GL}_p(\mathbb{Z}) \times \mathrm{GL}_q(\mathbb{Z}) \times \mathrm{GL}_{l_2}(\mathbb{Z})) \ltimes (\mathbb{Z}^r p \times \mathbb{Z}^r q \times \mathbb{Z}^{pql_2})$$

Since there are no other possibilities for the integers p, q and r, we have shown that the sequence $C\Gamma^k$ stabilizes after at most 5 steps, which completes the proof. □

As an immediate corollary of this Lemma and Theorem 4.5.2 we obtain the stabilization of the automorphism tower of the free nilpotent group $\Gamma(n,d)$ in the case when $\mathrm{Cen}(\mathrm{Aut}\,\Gamma)$ is not trivial.

THEOREM 4.5.5. *If $2n|d-1$, then The automorphism tower of the free nilpotent group $\Gamma(n,d)$ stabilizes after at most $\max\{5, 2+d/h(n)\}$ steps, in particular it has finite height.*

If we have that $d \leq h(n)$ we have that the tower stabilizes after at most 3 steps.

4.6. Two generated case

Finally let us consider the case $n = 2$ and $d = 4k+1$. In this case we can not apply directly the results from the previous section because $\mathrm{SL}_2(\mathbb{Z})$ has homologically nontrivial cocycles into $\mathrm{Cen}(UL)$ and therefore we can not embed the whole group $\mathrm{Aut}^2\,\Gamma$ into $\mathrm{Aut}\,G^1$, but we can embed very big part of it.

The automorphism group $\mathrm{Aut}\,G^1$ can be described in the same way as in Theorem 4.5.1 part a). Lets us obtain a similar description of the second group in the automorphism tower.

LEMMA 4.6.1. *The group $\mathrm{Aut}^2\,\Gamma$ contains a normal subgroup $A\Gamma^2$, which can be embedded into $\mathrm{Aut}\,G^1$. Also the factor group $\mathrm{Aut}^2\,\Gamma/A\Gamma^2$ is abelian. Also the group $A\Gamma^2$ slits as a direct product and has description similar to the one Theorem 4.5.1 b).*

PROOF. Let $A\Gamma^2$ be the subgroup of $\operatorname{Aut}^2\Gamma$ consisting of all automorphisms which can be extended to the group G^1, i.e., the cocycle $v : \mathrm{GL}_2(\mathbb{Z}) \to \mathrm{Cen}(UG)$ is homologically trivial. This group can be described as in Theorem 4.5.1 and therefore splits as a direct product. The factor group $\operatorname{Aut}^2\Gamma/A\Gamma^2$ is abelian because $H^1(\mathrm{GL}_2(\mathbb{Z}), \mathrm{Cen}\, UG)$ is abelian. $\square$

Let G^2 denote the Zariski closure of the group $A\Gamma^2$ in to $\operatorname{Aut} G^1$. This group has similar description to the one in Theorem 4.5.1.

We would like to use the group $A\Gamma^2$ to study the next group in the automorphism tower of Γ. Therefore we need to show that this is a characteristic subgroup of $\operatorname{Aut}^2\Gamma$.

LEMMA 4.6.2. *The subgroup $A\Gamma^2$ is preserved by all automorphisms of the group $\operatorname{Aut}^2\Gamma$.*

PROOF. Let B is the commutator subgroup of Γ. This is a subgroup of $A\Gamma^2$, and therefore can be embedded into some quotient G of G^2. Let us fix one such embedding $\phi : B \to G$.

Let U denote the unipotent radical of G and let $C = \phi^{-1}(\phi B \cap U)$ be the intersection of the group B with the radical of G. The rigidity Theorem 4.2.2 implies the the subgroup C does not depend on the choice of the embedding ϕ, since G has strong unipotent radical.

Every automorphism of the group B determines a cocycle form the semi simple part of B into the center of the unipotent radical of G. This gives us a map π form $\operatorname{Aut}^2\Gamma/B$ to $H^1(B/C, \mathrm{Cen}(U))$, because the conjugation with elements in $\operatorname{Aut}^2\Gamma/B$ gives automorphisms of the group B. Applying Theorem 4.2.2 again gives us that π does not depend on the choice of ϕ. Finally we can define $A\Gamma^2$ as $\pi^{-1}(0)$. Which shows that $A\Gamma^2$ is preserved by all automorphisms of the group $\operatorname{Aut}^2\Gamma$. $\square$

Combainning all this results we have shown that

THEOREM 4.6.3. *The second group* $\operatorname{Aut}^2\Gamma$ *in the automorphsim tower of* Γ, *contains a inraviant subgroup* $A\Gamma^2$, *which is a Zariski dense lattice int hte Lie group* G^2. *Also* $A\Gamma^2$ *splits as a direct product of* $N\Gamma^2$ *and* $C\Gamma^2$. *The groups* $N\Gamma^2$ *and* $C\Gamma^2$ *can be described in a way similar to Theorem 4.5.1. It can also be shown that the whole group* $\operatorname{Aut}^2\Gamma$ *splits as a direct product of* $\tilde{N}\Gamma^2$ *and* $C\Gamma^2$ *for some abelian extension* $\tilde{N}\Gamma^2$ *of* $N\Gamma^2$

PROOF. The only thing that does nnot follow from the previous lemmas is that the whole group $\operatorname{Aut}^2\Gamma$ splits as a direct product. This is true because there are no notrivial cocycles in the center of $\operatorname{Aut}\Gamma$, which implies that the conjugation with elements in $\operatorname{Aut}^2\Gamma$ acts trivially on the group $C\Gamma^2$. $\square$

As in the case of more than 2 generators, description of every group in the automorphism tower is similar to the one for the secon group. This allows us to generalize the previous Theorem to a description of the automorphism tower of the group Γ.

THEOREM 4.6.4. *a) Every group* $\operatorname{Aut}^k\Gamma$ *in the automorphism tower of* Γ, *contains a characteristic subgroup* $A\Gamma^k$ *such that all factors* $\operatorname{Aut}^k\Gamma/A\Gamma^k$ *for* $k \geq 2$ *are abelian and isomorphic to each other.*

b) Each group $A\Gamma^k$ *can be embedded as a lattice in a Lie group* G^k *and* $A\Gamma^{k+1}$ *can be defined as the subgroup of* $\operatorname{Aut} G^k$ *consisting of all automorphisms, which preserve the subgroup* $A\Gamma^k$.

c) Each group $A\Gamma^k$ *splits as a direct product of* $N\Gamma^k$ *and* $C\Gamma^k$ *where the group* $N\Gamma^k$ *has description similar to the ones in Theorem 4.5.2. Also the group* $\operatorname{Aut}^k\Gamma$ *splits as a direct product of* $\tilde{N}\Gamma^k$ *and* $C\Gamma^k$. *The group* $C\Gamma^k$ *are defined recursivly as*

follows

$$C\Gamma^{k+1} = \operatorname{Aut} C\Gamma^k \ltimes \operatorname{Hom}(\tilde{N}\Gamma^k/[\tilde{N}\Gamma^k, \tilde{N}\Gamma^k], \operatorname{Cen}(C\Gamma^k))$$

PROOF. The proof is by induction and uses result of Lemma 4.6.2 and Theorem 4.6.3 as a base case. The induction step is similar to the one in Theorem 4.5.2, but few additional things have to be shown. First, we need to show that $A\Gamma^k$ is a invarinat subgroup of $\operatorname{Aut}^k \Gamma$. The proof of this fact is the same as in the case $k = 2$ and we can repeat the argument from Lemma 4.6.2. All groups $\operatorname{Aut}^k \Gamma/A\Gamma^k$ are isomorphic because all nontrivial $\mathrm{GL}_2(\mathbb{Z})$ modules in the center of the nilpotent part of the group $A\Gamma^k$ comes from $\operatorname{Cen}(\mathrm{IAut}\,\Gamma)$.

Second, we need to mention that every automorphism of the group $\operatorname{Aut}^k \Gamma$ is almost determined by its action on $A\Gamma^k$. Using the fact that the adjoint action of $\operatorname{Aut}^k \Gamma/\operatorname{Cen}(\operatorname{Aut}^k \Gamma)$ is faithfull we can see that if $\phi \in \operatorname{Aut}(\operatorname{Aut}^k \Gamma)$ is such that the restriction $\phi_{|A\Gamma^k}$ of ϕ on $A\Gamma^k$ is the identity, then ϕ acts trivially on $\operatorname{Aut}^k \Gamma/\operatorname{Cen}(\operatorname{Aut}^k \Gamma)$. That is the reason why in the description of the group $C\Gamma^k$ we use the groups $\tilde{N}\Gamma^k$ not $N\Gamma^k$ as in Theorem 4.5.2.

Finally we need to mention that the whole group $\operatorname{Aut}^k \Gamma$ splits as a direct product because all the nontriviall cocylces of $\mathrm{GL}_2(\mathbb{Z})$ does not interesect the center of the group $\operatorname{Aut}^k \Gamma$. $\square$

As in the case of more than 2 generators, we have that the sequences $N\Gamma^k$ and $\tilde{N}\Gamma^k$ stabilize, since the sequence $\tilde{UL}_k$ stabilizes. In order to claim that the automorphism towers of the 2 generated nilpotent group stabilizes we need to show that the sequence $C\Gamma^k$ stabilizes.

We have that $C\Gamma^2$ is not trivial if and only if $d = 4k + 1$ for some positive integer k. If $k \geq 2$ then Lemma 4.5.3 holds, because the homogeneus component of degree

$4k$ of the free Lie algebra on 2 generators contains enough invarinat $\mathfrak{sl}_2$ modules. In this case and the stabilization of the sequnece $C\Gamma^k$ follows from Lemma 4.5.4

The case $d = 5$ have to be examined separately because there are no $\mathfrak{sl}_2$ invarinat modules in the homogeneous component of degree 4 in the free Lie algebra on 2 generators. In this case we have that $\mathrm{Cen}(\mathrm{Aut}\,\Gamma) = \mathbb{Z}$ and that $\mathrm{IAut}\,\Gamma^{[0]}_{\mathrm{gen}} = \mathbb{Z}$, therefore we have that $C\Gamma^2$ is either trivial or cyclic of order 2 (it is actually a trivial but to show this we need to examin how $\mathrm{Aut}\,\Gamma$ sitts inside G^1). If it is trivial then all groups $C\Gamma^k$ are trivial and the sequence stabilizes.

If it is not trivial we have to use the fact that

$$\tilde{N}\Gamma^2/[\tilde{N}\Gamma^2, \tilde{N}\Gamma^2] \simeq \mathbb{Z} \times \mathbb{F}_2,$$

where the $\mathbb{F}_2$ copes from the abealinization of $\mathrm{GL}_2(\mathbb{Z})$ and $\mathbb{Z}$ comes from the extension $\tilde{N}\Gamma^2$ over $N\Gamma^2$. Using this information we can see that $C\Gamma^3$ is elementary abelaing group of order 4 and that

$$C\Gamma^4 \simeq \mathrm{GL}_2(\mathbb{F}_2) \ltimes \mathrm{Hom}(\mathbb{F}_2^2, \mathbb{F}_2^2)$$

and that all other group are equal to

$$C\Gamma^k \simeq (\mathrm{GL}_2(\mathbb{F}_2) \times \mathrm{GL}_2(\mathbb{F}_2)) \ltimes \mathrm{Hom}(\mathbb{F}_2^2, \mathbb{F}_2^2)$$

for $k \geq 5$.

This prove that the sequence $C\Gamma^k$ stabilizes after at most 6 steps, which together with the stabilization of $\tilde{N}\Gamma^k$ finishes the proof of the following theorem

THEOREM 4.6.5. *The automorphism tower of the group* $\Gamma(2, d)$ *stabilizes after at most* $\max\{6, 2 + d/4\}$ *steps, if* $d = 4k + 1$.

This result together with Theorems 4.4.6 and 4.5.5, shows that the automorphim towers of the free nilpotent groups $\Gamma(n, d)$, stabilize after finetly many steps, for any values of n and d.

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